A finite element method with edge oriented stabilization for the time-dependent Navier-Stokes equations: space discretization and convergence

Erik Burman, Miguel Angel Fernández

INSTITUT NATIONAL DE RECHERCHE EN INFORMATIQUE ET EN AUTOMATIQUE

A finite element method with edge oriented stabilization for the time-dependent Navier-Stokes equations: space discretization and convergence

Erik Burman — Miguel Ángel Fernández

N° 5630

July 2005

Thème BIO

Rapport de recherche

ISSN 0249-6399 ISRN INRIA/RR--5630--FR+ENG

A finite element method with edge oriented stabilization for the time-dependent Navier-Stokes equations: space discretization and convergence

Erik Burman* , Miguel Ángel Fernández†

Thème BIO — Systèmes biologiques
Projet REO

Rapport de recherche n° 5630 — July 2005 — 42 pages

Abstract: This work focuses on the numerical analysis of a finite element method with edge oriented stabilization for the unsteady incompressible Navier-Stokes equations. Incompressibility and convective effects are both stabilized adding an interior penalty term giving L^2-control of the jump of the gradient of the approximate solution over the internal edges. Using continuous equal-order finite elements for both velocities and pressures, in a space semi-discretized formulation, we prove convergence of the approximate solution. The error estimates hold irrespective of the Reynolds number, and hence also for the incompressible Euler equations, provided the exact solution is smooth.

Key-words: finite element method, stabilized method, continuous interior penalty, Navier-Stokes equations, time-dependent problems.

Paper submitted to *Numerische Mathematik.*

* EPFL, IACS/CMCS, CH–1015 Lausanne, Switzerland. email: erik.burman@epfl.ch
† INRIA, projet REO. email: miguel.fernandez@inria.fr

Unité de recherche INRIA Rocquencourt
Domaine de Voluceau, Rocquencourt, BP 105, 78153 Le Chesnay Cedex (France)
Téléphone : +33 1 39 63 55 11 — Télécopie : +33 1 39 63 53 30

Une méthode d'éléments finis stabilisée par pénalisation intérieure pour les équations de Navier-Stokes instationnaires: discrétisation en espace et convergence

Résumé : Dans ce travail on s'intéresse à l'analyse numérique d'une méthode d'éléments finis stabilisée pour les équations de Navier-Stokes instationnaires. L'incompressibilité et la convection sont stabilisées en ajoutant un terme de pénalisation qui fournit un contrôle, en norme L^2, du saut du gradient de la solution sur les faces internes. En utilisant des éléments finis continus, du même ordre pour la vitesse et la pression et dans une formulation semi-discrétisée en espace, on montre convergence de l'approximation. Les estimations d'erreur sont indépendantes du nombre de Reynolds (et donc valables pour le cas des équations d'Euler) en supposant que la solution est suffisamment régulière.

Mots-clés : méthode d'éléments finis, méthode stabilisée, pénalisation intérieure, équations de Navier-Stokes, problème instationnaire.

1 Introduction

In this paper we propose a finite element method based on edge oriented stabilization for the incompressible Navier-Stokes equations. This method method was introduced by Burman and Hansbo in [11], as an extension of the interior penalty method proposed by Douglas and Dupont in [17] to the case of pure transport problems or convection–dominated problems. Pressure stabilization for the Stokes problem was then considered by Burman and Hansbo in [12] and the Oseen's problem was analyzed by Burman, Fernández and Hansbo in [10]. In the latter, a priori error estimates that hold uniformly in the Reynolds number were proven for sufficiently smooth solutions. In this paper we focus on the time dependent, non-linear Navier-Stokes equations. There exists a vast literature on finite element methods for the Navier-Stokes equations. Let us cite the monograph of Girault and Raviart [20] and the series of papers by Heywood and Rannacher [25, 26, 27]. In the case of stabilized finite elements using SUPG-like stabilizations, we cite the work of Johnson and Saranen [29] on a velocity-vorticity formulation, and the paper by Hansbo and Szepessy on the velocity-pressure formulation [23]. Other relevant works on the Navier-Stokes equations include the paper by Tobiska and Verführt [32], the work by Blasco and Codina [15], the work on stabilized mixed methods for the Navier-Stokes equations by He, Lin and Sun [24], and the work on numerical methods for LES using hyperviscosity by Guermond and Prud'homme [22]. For relevant references on stabilized methods we refer to the subgrid viscosity method by Guermond [21], the orthogonal subscale method by Codina [14], the local projection method by Becker, Braack and Burman [1, 2, 5] and the work on minimal stabilisation procedures by Brezzi and Fortin [8].

The key issue in this paper is that the stabilization allows for estimates that are uniform in the Reynolds number. Hence the incompressible Euler equations are covered by the analysis. It is interesting to note that the present stabilized method allows for a complete decoupling of the analysis for the velocities and pressures. The only requirement for convergence is that the solution is sufficiently smooth, in a sense that will be detailed later, but most importantly we assume that the velocities $\mathbf{u} \in [L^2(0,T;H^{\frac{3}{2}+\epsilon}(\Omega)) \cap L^\infty(0,T;W^{1,\infty}(\Omega)) \cap H^1(0,T;L^2(\Omega))]^d$ and the pressure $p \in L^2(0,T;H^{\frac{1}{2}+\epsilon}(\Omega))$. In case the solution has sufficient additional regularity we obtain the quasi optimal error estimate for the velocity approximation:

$$\|\mathbf{u}-\mathbf{u}_h\|_{L^\infty(0,T;L^2(\Omega))} \leq Ch^{k+\frac{1}{2}}\|(\mathbf{u},p)\|_{L^2(0,T;H^{k+1}(\Omega))},$$

where k denotes the polynomial order.

Our analysis is inspired by the one by Hansbo and Szepessy reported in [23], but our results using edge oriented stabilization are sharper. In fact, to control the convective velocity, which is only weakly divergence free, special nonlinear stabilization terms are introduced in [23], leading to a more complex formulation and stronger regularity assumptions on the exact solution are required. In our case, the fact that the stabilization of the velocities is decoupled from the stabilization of the pressure allows us to prove convergence using essentially the stabilization terms of the linear case (see [10]), and under similar regularity assumptions. Moreover, we prove convergence for all polynomial orders, whereas in [23] the analysis was restricted to piecewise linear approximations in space and in time.

In this work we only consider discretization in space. Focus will be put entirely on the convergence in the high Reynolds number regime ($\nu < h$). The estimates are of course still valid in the low Reynolds number regime ($h < \nu$), but then the regularity hypothesis may be relaxed while keeping optimal convergence if the stabilization parameters are properly chosen, see [10].

In the next section we introduce standard notation for the Navier-Stokes equations and briefly discuss the regularity assumptions. The stabilized finite element scheme, based on an interior penalty formulation, is introduced in section 3. Some useful standard estimates are stated in section 4. In section 5, we study the wellposedness of the discrete scheme and its stability properties. The convergence analysis of the method is carried out in section 6. First we prove convergence for the velocity and then for the pressure. The later requires an estimation of the error in the approximate acceleration. Finally, some conclusions are given in section 7.

2 The time-dependent Navier-Stokes equations

Let Ω be a Lipschitz-continuous domain in $\mathbb{R}^d$ ($d = 2$ or 3) with a polyhedral boundary $\partial\Omega$ and outward pointing normal $\mathbf{n}$. For $T > 0$ we consider the problem of solving, for $\mathbf{u} : \Omega \times (0, T) \longrightarrow \mathbb{R}^d$ and $p : \Omega \times (0, T) \longrightarrow \mathbb{R}$, the time-dependent incompressible Navier-Stokes equations with homogeneous boundary conditions (for the sake of simplicity):

$$\left\{\begin{aligned} \partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} - 2\nu \nabla \cdot \boldsymbol{\varepsilon}(\mathbf{u}) + \nabla p = \mathbf{f} \quad & \text{in} \quad \Omega \times (0, T), \\ \nabla \cdot \mathbf{u} = 0 \quad & \text{in} \quad \Omega \times (0, T), \\ \mathbf{u} = 0 \quad & \text{on} \quad \partial\Omega \times (0, T), \\ \mathbf{u}(\cdot, 0) = \mathbf{u}_0 \quad & \text{in} \quad \Omega. \end{aligned}\right. \tag{1}$$

These equations describe the motion of a viscous incompressible fluid confined in Ω. In (1), $\nu > 0$ corresponds to the kinematic fluid viscosity coefficient, $\mathbf{f} : \Omega \times (0, T) \longrightarrow \mathbb{R}^d$ represents a given source term, $\mathbf{u}_0 : \Omega \longrightarrow \mathbb{R}^d$ stands for the initial velocity and

$$\boldsymbol{\varepsilon}(\mathbf{u}) \stackrel{\text{def}}{=} \frac{1}{2}\left[\nabla \mathbf{u} + (\nabla \mathbf{u})^{\mathrm{T}})\right],$$

for the strain rate tensor.

In the following, we will consider the usual Sobolev spaces $W^{m,q}(\Omega)$, with norm $\|\cdot\|_{m,q,\Omega}$, $m \geq 0$ and $q \geq 1$. In particular, we have $L^q(\Omega) = W^{0,q}(\Omega)$. We use the standard notation $H^m(\Omega) \stackrel{\text{def}}{=} W^{m,2}(\Omega)$. The norm of $H^m(\Omega)$ is denoted by $\|\cdot\|_{m,\Omega}$ and its semi-norm by $|\cdot|_{m,\Omega}$. The space of $L^2(\Omega)$ divergence free functions is denoted by $H_0(\mathrm{div}; \Omega)$. The scalar product in $L^2(\Omega)$ is denoted by $(\cdot, \cdot)$ and its norm by $\|\cdot\|_{0,\Omega}$. The closed subspaces $H_0^1(\Omega)$, consisting of functions in $H^1(\Omega)$ with zero trace on $\partial\Omega$, and $L_0^2(\Omega)$, consisting of function in $L^2(\Omega)$ with zero mean in Ω, will also be used.

Let us assume that the given functions $\mathbf{f}$ and $\mathbf{u}_0$ have, at least, the following regularity properties

$$\mathbf{f} \in L^\infty(0,T;[L^2(\Omega)]^d), \quad \mathbf{u}_0 \in [L^2(\Omega)]^d.$$

For sufficiently regular functions $\mathbf{u}$ and p, problem (1) holds if and only if

$$\begin{cases} (\partial_t \mathbf{u}, \mathbf{v}) + c(\mathbf{u};\mathbf{u},\mathbf{v}) + a(\mathbf{u},\mathbf{v}) + b(p,\mathbf{v}) = (\mathbf{f},\mathbf{v}), & \text{a.e. in} \quad (0,T) \\ b(q,\mathbf{u}) = 0, & \text{a.e. in} \quad (0,T), \\ \mathbf{u}(0) = \mathbf{u}_0, & \text{a.e. in} \quad \Omega, \end{cases} \tag{2}$$

for all $(\mathbf{v},q) \in [H^1_0(\Omega)]^d \times L^2_0(\Omega)$, and where

$$\begin{aligned} c(\mathbf{w};\mathbf{u},\mathbf{v}) &\stackrel{\text{def}}{=} (\mathbf{w}\cdot\nabla\mathbf{u},\mathbf{v}), \\ a(\mathbf{u},\mathbf{v}) &\stackrel{\text{def}}{=} 2(\nu\boldsymbol{\varepsilon}(\mathbf{u}),\boldsymbol{\varepsilon}(\mathbf{v})), \\ b(p,\mathbf{v}) &\stackrel{\text{def}}{=} -(p,\nabla\cdot\mathbf{v}). \end{aligned} \tag{3}$$

2.1 Regularity assumptions

For the analysis below to make sense, the solution and initial data must have the minimal regularity

$$\begin{aligned} &\mathbf{u} \in [L^2(0,T;H^{\frac{3}{2}+\epsilon}(\Omega)) \cap L^\infty(0,T;W^{1,\infty}(\Omega)) \cap H^1(0,T;L^2(\Omega))]^d, \\ &p \in L^2(0,T;H^{\frac{1}{2}+\epsilon}(\Omega)), \quad \mathbf{u}_0 \in [H^{\frac{3}{2}+\epsilon}(\Omega)\cap H^1_0(\Omega)]^d \cap H_0(\mathrm{div};\Omega). \end{aligned} \tag{4}$$

In this paper we will for simplicity make the stronger regularity assumption

$$\begin{aligned} &\mathbf{u} \in [L^\infty(0,T;W^{1,\infty}(\Omega)) \cap H^1(0,T;L^2(\Omega)) \cap L^2(0,T;H^r(\Omega))]^d, \\ &p \in L^2(0,T;H^s(\Omega)), \quad \mathbf{u}_0 \in [H^r(\Omega)\cap H^1_0(\Omega)]^d \cap H_0(\mathrm{div};\Omega). \end{aligned} \tag{5}$$

with $r, s \geq 2$, in order to use approximability and get optimal order estimates for the velocity.

Our pressure error estimates are bounded by the L^2-norm of the error in the approximate acceleration $\partial_t \mathbf{u}_h$. The error estimate we provide for this quantity requires the following additional regularity

$$\begin{aligned} &\mathbf{u} \in [H^1(0,T;H^r(\Omega))]^d, \\ &p \in L^2(0,T;H^s(\Omega)) \cap H^1(0,T;H^1(\Omega)). \end{aligned} \tag{6}$$

3 Space semi-discretization

In this section we introduce a finite element discretization of problem (2) based on a strongly consistent interior penalty formulation with equal-order interpolations.

3.1 Preliminaries

Let $\{\mathcal{T}_h\}_{0<h\leq 1}$ a family of triangulations of the domain Ω without hanging nodes. For each triangulation $\mathcal{T}_h$, the subscript $h \in (0,1]$ refers to the level of refinement of the triangulation, which is defined by

$$h \stackrel{\text{def}}{=} \max_{K\in\mathcal{T}_h} h_K, \quad h_K \stackrel{\text{def}}{=} \max_{e\subset\partial K} h_e,$$

with h_e the diameter of the face e.

Moreover we will assume that the family of triangulation $\{\mathcal{T}_h\}_{0<h\leq 1}$ is quasi-uniform, i.e.,

$$\frac{h_K}{\rho_K} < C_{\mathrm{R}}, \quad h_K \geq C_{\mathrm{U}} h, \quad \forall K \in \mathcal{T}_h, \quad \forall h \in (0,1], \tag{7}$$

where ρ_K stands for the diameter of the largest inscribed ball in K and $C_{\mathrm{R}}, C_{\mathrm{U}} > 0$ are fixed constants.

In the sequel, the word faces refers to edges in 2D and faces in 3D, and the distinction will not be made unless necessary. For a given piecewise continuous function φ, the jump $[\![\varphi]\!]_e$ over a face e is defined by

$$[\![\varphi]\!]_e(\mathbf{x}) \stackrel{\text{def}}{=} \begin{cases} \lim\limits_{t\to 0^+} \big(\varphi(\mathbf{x}-t\mathbf{n}_e) - \varphi(\mathbf{x}+t\mathbf{n}_e)\big), & \text{if} \quad e \not\subset \partial\Omega, \\ 0, & \text{if} \quad e \subset \partial\Omega, \end{cases}$$

where $\mathbf{n}_e$ is a normal unit vector on e and $\mathbf{x} \in e$.

In this paper, we let V_h^k denote the standard space of continuous piecewise polynomial functions of degree k,

$$V_h^k \stackrel{\text{def}}{=} \left\{ v \in H^1(\Omega) \,:\, v_{|K} \in \mathbb{P}_k(K), \quad \forall K \in \mathcal{T}_h \right\},$$

and $H^2(\mathcal{T}_h)$ the space of piecewise H^2 functions,

$$H^2(\mathcal{T}_h) \stackrel{\text{def}}{=} \left\{ v : \Omega \longrightarrow \mathbb{R} \,:\, v_{|K} \in H^2(K), \quad \forall K \in \mathcal{T}_h \right\}. \tag{8}$$

For the velocities we will use the space $[V_h^k]^d$ and for the pressure we will use $Q_h^k \stackrel{\text{def}}{=} V_h^k \cap L_0^2(\Omega)$.

3.2 An interior penalty finite element method

Denoting the product space $W_h^k \stackrel{\text{def}}{=} [V_h^k]^d \times Q_h^k$ our space semi-discretized scheme reads: for all $t \in (0,T)$, find $(\mathbf{u}_h(t), p_h(t)) \in W_h^k$ such that

$$\begin{aligned} (\partial_t \mathbf{u}_h, \mathbf{v}_h) + (\mathbf{A}+\mathbf{J})\big[\mathbf{u}_h; (\mathbf{u}_h, p_h), (\mathbf{v}_h, q_h)\big] &= (\mathbf{f}, \mathbf{v}_h), \\ \mathbf{u}_h(0) &= \mathbf{u}_{0,h}, \end{aligned} \tag{9}$$

for all $(\mathbf{v}_h, q_h) \in W_h^k$ and with $\mathbf{u}_{0,h}$ a suitable approximation of $\mathbf{u}_0$ in $[V_h^k]^d$. In (9) we used the following notations:

$$\begin{aligned}\mathbf{A}\big[\mathbf{w}_h; (\mathbf{u}_h, p_h), (\mathbf{v}_h, q_h)\big] \stackrel{\text{def}}{=} & \; a_h(\mathbf{u}_h, \mathbf{v}_h) + c_h(\mathbf{w}_h; \mathbf{u}_h, \mathbf{v}_h) \\ & + b_h(p_h, \mathbf{v}_h) - b_h(q_h, \mathbf{u}_h),\end{aligned} \tag{10}$$

$$\begin{aligned}c_h(\mathbf{w}_h; \mathbf{u}_h, \mathbf{v}_h) \stackrel{\text{def}}{=} & \; c(\mathbf{w}_h; \mathbf{u}_h, \mathbf{v}_h) + \frac{1}{2}(\nabla \cdot \mathbf{w}_h \mathbf{u}_h, \mathbf{v}_h) \\ & - \frac{1}{2}\left\langle \mathbf{w}_h \cdot \mathbf{n} \mathbf{u}_h, \mathbf{v}_h \right\rangle_{\partial\Omega},\end{aligned} \tag{11}$$

$$\begin{aligned}a_h(\mathbf{u}_h, \mathbf{v}_h) \stackrel{\text{def}}{=} & \; a(\mathbf{u}_h, \mathbf{v}_h) - \left\langle 2\nu\boldsymbol{\varepsilon}(\mathbf{u}_h)\mathbf{n}, \mathbf{v}_h \right\rangle_{\partial\Omega} - \left\langle \mathbf{u}_h, 2\nu\boldsymbol{\varepsilon}(\mathbf{v}_h)\mathbf{n} \right\rangle_{\partial\Omega} \\ & + \left\langle \gamma_\nu \frac{\nu}{h} \mathbf{u}_h, \mathbf{v}_h \right\rangle_{\partial\Omega} + \left\langle \mathbf{u}_h \cdot \mathbf{n}, \mathbf{v}_h \cdot \mathbf{n} \right\rangle_{\partial\Omega},\end{aligned} \tag{12}$$

$$b_h(p_h, \mathbf{v}_h) \stackrel{\text{def}}{=} b(p_h, \mathbf{v}_h) + \left\langle p_h, \mathbf{v}_h \cdot \mathbf{n} \right\rangle_{\partial\Omega}, \tag{13}$$

$$\begin{aligned}\mathbf{J}\big[\mathbf{w}_h; (\mathbf{u}_h, p_h), (\mathbf{v}_h, q_h)\big] \stackrel{\text{def}}{=} & \; j_{\mathbf{w}_h}(\mathbf{u}_h, \mathbf{v}_h) + \gamma j(\mathbf{u}_h, \mathbf{v}_h) \\ & + j(p_h, q_h),\end{aligned} \tag{14}$$

with

$$\begin{aligned}j_{\mathbf{w}_h}(\mathbf{u}_h, \mathbf{v}_h) & \stackrel{\text{def}}{=} \sum_{K \in \mathcal{T}_h} \int_{\partial K} h_K^2 |\mathcal{I}_h^1 \mathbf{w}_h \cdot \mathbf{n}|^2 [\![\nabla \mathbf{u}_h]\!] : [\![\nabla \mathbf{v}_h]\!] \, \mathrm{d}\boldsymbol{s}, \\ j(\mathbf{u}_h, \mathbf{v}_h) & \stackrel{\text{def}}{=} \sum_{K \in \mathcal{T}_h} \int_{\partial K} h_K^2 [\![\nabla \mathbf{u}_h]\!] : [\![\nabla \mathbf{v}_h]\!] \, \mathrm{d}\boldsymbol{s}, \\ j(p_h, q_h) & \stackrel{\text{def}}{=} \sum_{K \in \mathcal{T}_h} \int_{\partial K} h_K^2 [\![\nabla p_h]\!] \cdot [\![\nabla q_h]\!] \, \mathrm{d}\boldsymbol{s}.\end{aligned} \tag{15}$$

Here, $\mathcal{I}_h^1 \mathbf{w}_h$ denotes the interpolation of $\mathbf{w}_h$ onto the space $[V_h^1]^d$ (continuous piecewise linear) and γ, γ_ν two positive constants to be fixed later on.

Some remarks are in order. We point out that the additional terms appearing in the discrete bilinear form $\mathbf{A}$, compared to the formulation (2), are due to the non satisfaction of the divergence free condition and to the weakly imposed boundary conditions of Nitsche type. To counter effects of insufficient control of the divergence free condition, an artificial term is added that ensures coercivity while remaining strongly consistent (since $\nabla \cdot \mathbf{u} = 0$ for the exact solution). The Nitsche type boundary conditions are inspired by those analyzed in [10] and [19]. In the stabilization term $j_{\mathbf{w}_h}(\cdot, \cdot)$ we use the $\mathbb{P}_1$-interpolant of the velocity vector $\mathbf{w}_h$ as weight. This may be replaced by the function $\mathbf{w}_h$ itself or the max value of $\mathbf{w}_h$ on the face depending on what is most convenient from implementation standpoint. The analysis below carries over to these versions with minor modifications.

The discrete formulation (9) satisfies the following approximate Galerkin Orthogonality.

Lemma 3.1 (Approximate Galerkin Orthogonality) *Let* $(\mathbf{u}, p)$ *the solution of* (1), $(\mathbf{u}_h, p_h) \in W_h^k$ *the solution of* (9) *and assume that* $(\mathbf{u}, p)$ *has the minimal regularity* (4). *Then,*

$$(\partial_t(\mathbf{u} - \mathbf{u}_h), \mathbf{v}_h) + \mathbf{A}\big[\mathbf{u}; (\mathbf{u}, p), (\mathbf{v}_h, q_h)\big] \\ - (\mathbf{A} + \mathbf{J})\big[\mathbf{u}_h, (\mathbf{u}_h, p_h), (\mathbf{v}_h, q_h)\big] = 0, \quad a.e.\ in \quad (0, T),$$

for all $(\mathbf{v}_h, q_h) \in W_h^k$.

Proof. This is an immediate consequence of the consistency of the standard Galerkin method. □

4 Interpolation

In this section we shall state some standard estimates that will be useful for the convergence analysis below. First, we recall the following local inverse estimate (see [18, page 75], for instance): for all $v_h \in V_h^k$, and $K \in \mathcal{T}_h$, $0 < h \leq 1$, there holds

$$\|v_h\|_{l,p,K} \leq C_{\mathrm{I}} h_K^{m-l+d\left(\frac{1}{p}-\frac{1}{q}\right)} \|v_h\|_{m,q,K}, \tag{16}$$

with C_{I} a positive constant, independent of h, K, p and q, and where $0 \leq m \leq l$ and $1 \leq p, q \leq \infty$.

Let Π_h^k and $\mathcal{I}_h^k$ be, respectively, the L^2-projection and the Lagrange interpolant on V_h^k. For $u \in H^r(\Omega)$, $r \geq 2$, we have the following standard error estimate (see [18], for instance),

$$\|\mathcal{I}_h^k u - u\|_{0,\Omega} + h\|\nabla(\mathcal{I}_h^k u - u)\|_{0,\Omega} \leq C h^{r_{\mathbf{u}}} \|u\|_{r_{\mathbf{u}},\Omega}, \tag{17}$$

where $r_{\mathbf{u}} = \min(r, k+1)$. The following stability estimates for the L^2-projection hold,

$$\begin{aligned} \|\Pi_h^k u\|_{0,\Omega} &\leq C\|u\|_{0,\Omega}, \\ \|\Pi_h^k u\|_{1,\Omega} &\leq C\|u\|_{1,\Omega}, \end{aligned} \tag{18}$$

for all $u \in H^1(\Omega)$. Thus, from (17), we then deduce that

$$\|u - \Pi_h^k u\|_{0,\Omega} + h\|\nabla(u - \Pi_h^k u)\|_{0,\Omega} \leq C h^{r_{\mathbf{u}}} \|u\|_{r_{\mathbf{u}},\Omega}, \tag{19}$$

for $u \in H^r(\Omega)$. In addition, the following stability result holds where $C_\pi > 0$ is a constant independent of h (but not of the polynomial order),

$$\|\Pi_h^k u\|_{0,\infty,\Omega} \leq C_\pi \|u\|_{0,\infty,\Omega}, \quad \forall u \in L^\infty(\Omega), \tag{20}$$

$$\|\Pi_h^k u\|_{1,\infty,\Omega} \leq C_\pi \|u\|_{1,\infty,\Omega}, \quad \forall u \in W^{1,\infty}(\Omega). \tag{21}$$

The second estimate easily follows from the first noting that, from (16), we have

$$\begin{aligned}
\|\nabla\Pi_h^k u\|_{0,\infty,\Omega} &= \|\nabla(\Pi_h^k u - \Pi_{\tilde K}^0 u)\|_{0,\infty,\tilde K} \\
&\le Ch^{-1}\|\Pi_h^k u - \Pi_{\tilde K}^0 u\|_{0,\infty,\tilde K} \\
&\le Ch^{-1}\left(\|\Pi_h^k u - u\|_{0,\infty,\Omega} + \|u - \Pi_{\tilde K}^0 u\|_{0,\infty,\tilde K}\right) \\
&\le Ch^{-1}\left(\|\mathcal{I}_h^k u - u\|_{0,\infty,\Omega} + \|u - \Pi_{\tilde K}^0 u\|_{0,\infty,\tilde K}\right),
\end{aligned}$$

where $\tilde K \in \mathcal{T}_h$ stands for the element where the maximum value is taken, and $\Pi_{\tilde K}^0 u$ denotes the L^2-projection of u onto a piecewise constant on $\tilde K$. Applying now (20) to the first term of the right hand side we conclude

$$\begin{aligned}
\|\nabla\Pi_h^k u\|_{0,\infty,\Omega} &\le Ch^{-1}\left(\|\mathcal{I}_h^k u - u\|_{0,\infty,\Omega} + \|u - \Pi_{\tilde K}^0 u\|_{0,\infty,\tilde K}\right) \\
&\le C\|\nabla u\|_{0,\infty,\Omega}.
\end{aligned}$$

It then follows that

$$\|\Pi_h^k u - u\|_{0,\infty,\Omega} + h\|\Pi_h^k u - u\|_{1,\infty,\Omega} \le Ch\|u\|_{1,\infty,\Omega}, \tag{22}$$

for all $u \in W^{1,\infty}(\Omega)$.

The results (20) and (21) have been proved in [16, 6, 4] for low order elements. We would like to point to the last reference, which readily extends this results to higher order elements and which gives weighted estimates. Using these estimates, the assumption of mesh quasi-uniformity in the present paper may be relaxed to local quasi-uniformity.

In order to handle the non-linear terms, we shall also need a discrete commutator property, which is stated in the following lemma (for a proof, see [3]).

Lemma 4.1 *Let $\mathcal{SZ}_h^k : W^{1,\infty}(\Omega) \to V_h^k$ the Scott-Zhang interpolator [31]. There exists a constant $C_{\mathrm{B}} > 0$ independent of h, such that for all $u \in W^{1,\infty}(\Omega)$ and $v_h \in V_h^k$,*

$$\|\mathcal{SZ}_h^k(uv_h) - uv_h\|_{0,\Omega} \le C_B h\|u\|_{1,\infty,\Omega}\|v_h\|_{0,\Omega}.$$

The following corollary is a direct consequence of the previous result.

Corollary 4.1 *For all $u \in W^{1,\infty}(\Omega)$ and $v_h \in V_h^k$ there holds*

$$\begin{aligned}
\|\Pi_h^k(uv_h) - uv_h\|_{0,\Omega} &\le h\|u\|_{1,\infty,\Omega}\|v_h\|_{0,\Omega}, \\
\|\Pi_h^k(uv_h) - \Pi_h^k uv_h\|_{0,\Omega} &\le h\|u\|_{1,\infty,\Omega}\|v_h\|_{0,\Omega}.
\end{aligned} \tag{23}$$

Proof. Using the stability of the L^2-projection we may write

$$\begin{aligned}
\|\Pi_h^k(uv_h) - uv_h\|_{0,\Omega} &\le \|\Pi_h^k(uv_h) - \mathcal{SZ}_h(uv_h)\|_{0,\Omega} \\
&\quad + \|\mathcal{SZ}_h(uv_h) - uv_h\|_{0,\Omega} \\
&\le \|\Pi_h^k(uv_h - \mathcal{SZ}_h(uv_h))\|_{0,\Omega} \\
&\quad + \|\mathcal{SZ}_h(uv_h) - uv_h\|_{0,\Omega} \\
&\le 2\|\mathcal{SZ}_h(uv_h) - uv_h\|_{0,\Omega}.
\end{aligned}$$

Thus, $(23)_1$ holds from Lemma 4.1. Finally, for the second estimate we have

$$\begin{aligned}\|\Pi_h^k(uv_h) - \Pi_h^k u v_h\|_{0,\Omega} &\leq \|\Pi_h^k(uv_h) - uv_h\|_{0,\Omega} \\ &\quad + \|(u - \Pi_h^k u)\cdot v_h\|_{0,\Omega} \\ &\leq \|\Pi_h^k(uv_h) - uv_h\|_{0,\Omega} \\ &\quad + \|u - \Pi_h^k u\|_{0,\infty,\Omega}\|v_h\|_{0,\Omega}.\end{aligned}$$

Thus, $(23)_2$ is obtained by combining $(23)_1$ with (22). □

For the error analysis, we shall also use the trace inequality

$$\|v\|_{0,\partial K}^2 \leq C\Big(h_K^{-1}\|v\|_{0,K}^2 + h_K\|\nabla v\|_{0,K}^2\Big), \quad \forall v \in H^1(K), \tag{24}$$

see [13] for a proof. In particular, by combining the above estimate with a inverse inequality (16), it follows that

$$\|v_h\|_{0,\partial K}^2 \leq C_{\mathrm{T}} h_K^{-1}\|v_h\|_{0,K}^2, \quad \forall v_h \in V_h^k. \tag{25}$$

The uniform (in ν) stability of the present method relies on the fact that the gradient jumps in (9) can control some interpolation errors of the stream-line derivative, divergence and pressure gradient. This is formalized, in the following lemma, by establishing some error bounds for the Oswald quasi-interpolant π_h^* (see [28, 30]).

Definition 4.1 *For each node x_i, let n_i be the number of elements containing x_i as a node. We define a quasi-interpolant π_h^* of degree k by*

$$\pi_h^* v(x_i) \stackrel{\text{def}}{=} \frac{1}{n_i}\sum_{\{K\,:\,x_i\in K\}} v_{|K}(x_i), \quad \forall v \in [H^2(\mathcal{T}_h)]^d,$$

with $H^2(\mathcal{T}_h)$ given by (8).

Lemma 4.2 *There exist three constants $\gamma_i > 0$, $i = 1,2,3$, depending on the local mesh geometry, but not on the mesh size h, such that*

$$\|h^{\frac{1}{2}}\big(\mathbf{w}_h\cdot\nabla\mathbf{v}_h - \pi_h^*(\mathbf{w}_h\cdot\nabla\mathbf{v}_h)\big)\|_{0,\Omega}^2 \leq \gamma_1 j_{\mathbf{w}_h}(\mathbf{v}_h,\mathbf{v}_h), \tag{26}$$

$$\|h^{\frac{1}{2}}\big(\nabla\cdot\mathbf{v}_h - \pi_h^*(\nabla\cdot\mathbf{v}_h)\big)\|_{0,\Omega}^2 \leq \gamma_2 j(\mathbf{v}_h,\mathbf{v}_h), \tag{27}$$

$$\|h^{\frac{1}{2}}\big(\nabla q_h - \pi_h^*(\nabla q_h)\big)\|_{0,\Omega}^2 \leq \gamma_3 j(q_h,q_h), \tag{28}$$

for all $(\mathbf{v}_h, q_h, \mathbf{w}_h) \in [V_h^k]^d \times V_h^k \times [V_h^1]^d$.

Proof. A proof of (26)-(28) can be found in [9, 10]. □

We introduce now, for each $\mathbf{w}_h \in [V_h^k]^d$ given, the triple-norm

$$|\!|\!|(\mathbf{v}_h, q_h)|\!|\!|_{\mathbf{w}_h}^2 \stackrel{\text{def}}{=} \|\mathbf{v}_h\|^2 + \mathbf{J}\big[\mathbf{w}_h;(\mathbf{v}_h,q_h),(\mathbf{v}_h,q_h)\big], \tag{29}$$

with

$$\|\mathbf{v}_h\|^2 \stackrel{\text{def}}{=} \|\nu^{\frac{1}{2}}\nabla\mathbf{v}_h\|_{0,\Omega}^2 + \|h^{\frac{1}{2}}\nabla\cdot\mathbf{v}_h\|_{0,\Omega}^2 + \|(\gamma_\nu\nu)^{\frac{1}{2}}h^{-\frac{1}{2}}\mathbf{v}_h\|_{0,\partial\Omega}^2 + \|\mathbf{v}_h\cdot\mathbf{n}\|_{0,\partial\Omega}^2.$$

For the continuity of the Stokes system, with Nitsche boundary conditions, it is also convenient to introduce a norm valid for functions $(\mathbf{v},q)\in[H^{\frac{3}{2}+\epsilon}(\Omega)]^d\times H^{\frac{1}{2}+\epsilon}(\Omega)$,

$$\begin{aligned}|\!|\!|(\mathbf{v},q)|\!|\!|^2 \stackrel{\text{def}}{=} {}& \|\nu^{\frac{1}{2}}\nabla\mathbf{v}\|_{0,\Omega}^2 + \|h^{\frac{1}{2}}\nabla\cdot\mathbf{v}\|_{0,\Omega}^2 + \|h^{-\frac{1}{2}}q\|_{0,\Omega}^2\\ &+ \|h^{-\frac{1}{2}}\mathbf{v}\|_{0,\Omega} + \|(\nu h)^{\frac{1}{2}}\nabla\mathbf{v}\|_{0,\partial\Omega}^2 + \|q\|_{0,\partial\Omega}^2.\end{aligned} \tag{30}$$

For these two norms we have the following approximation result.

Lemma 4.3 *Assume that* (5) *holds. Then we have*

$$\begin{aligned}|\!|\!|(\mathbf{u}-\Pi_h^k\mathbf{u},p-\Pi_h^k p)|\!|\!|_{\mathbf{0}} \le {}& C(\nu^{\frac{1}{2}}+h^{\frac{1}{2}})h^{r_{\mathbf{u}}-1}\|\mathbf{u}\|_{r_{\mathbf{u}},\Omega}\\ &+ Ch^{r_p-\frac{1}{2}}\|p\|_{r_p,\Omega},\end{aligned} \tag{31}$$

and

$$\begin{aligned}|\!|\!|(\mathbf{u}-\Pi_h^k\mathbf{u},p-\Pi_h^k p)|\!|\!| \le {}& C(\nu^{\frac{1}{2}}+h^{\frac{1}{2}})h^{r_{\mathbf{u}}-1}\|\mathbf{u}\|_{r_{\mathbf{u}},\Omega}\\ &+ Ch^{r_p-\frac{1}{2}}\|p\|_{r_p,\Omega},\end{aligned} \tag{32}$$

with $r_{\mathbf{u}}=\min(r,k+1)$ *and* $r_p=\min(s,k+1)$, $C>0$ *a constant depending only on* γ_ν ,γ.

Proof. From (19) we have

$$\|\nu^{\frac{1}{2}}\nabla(\mathbf{u}-\Pi_h^k\mathbf{u})\|_{0,\Omega}^2 \le C\nu h^{2(r_{\mathbf{u}}-1)}\|\mathbf{u}\|_{r_{\mathbf{u}},\Omega}^2,$$

and

$$\|h^{\frac{1}{2}}\nabla\cdot(\mathbf{u}-\Pi_h^k\mathbf{u})\|_{0,\Omega}^2 \le Ch^{2r_{\mathbf{u}}-1}\|\mathbf{u}\|_{r_{\mathbf{u}},\Omega}^2.$$

We treat the boundary terms using the trace inequality (24) in combination with (19) and the quasi-uniformity of the triangulation (7), yielding

$$\begin{aligned}\|\mathbf{u}-\Pi_h^k\mathbf{u}\|_{0,\partial\Omega}^2 &\le C\sum_{e\subset\partial\Omega}\big(h_{K_e}^{-1}\|\mathbf{u}-\Pi_h^k\mathbf{u}\|_{0,K_e}^2\\ &\quad + h_{K_e}\|\nabla(\mathbf{u}-\Pi_h^k\mathbf{u})\|_{0,K_e}^2\big)\\ &\le Ch^{2r_{\mathbf{u}}-1}\|\mathbf{u}\|_{r_{\mathbf{u}},\Omega}^2,\end{aligned} \tag{33}$$

where K_e denotes the simplex such that $e \subset \partial K_e \cap \partial\Omega$. The interior penalty terms are treated in the same fashion as the boundary terms. We have

$$
\begin{aligned}
j(\mathbf{u} - \Pi_h^k \mathbf{u}, \mathbf{u} - \Pi_h^k \mathbf{u}) &= \sum_{K\in\mathcal{T}_h} h_K^2 \int_{\partial K} [\![\nabla(\mathbf{u} - \Pi_h^k \mathbf{u})]\!]^2 \,\mathrm{d}\boldsymbol{s} \\
&\leq C \sum_{K\in\mathcal{T}_h} h_K^2 \|\nabla(\mathbf{u} - \Pi_h^k \mathbf{u})\|_{0,\partial K}^2 \\
&\leq C \sum_{K\in\mathcal{T}_h} \left(h_K \|\nabla(\mathbf{u} - \Pi_h^k \mathbf{u})\|_{0,K}^2 + h_K^3 \|\nabla^2(\mathbf{u} - \Pi_h^k \mathbf{u})\|_{0,K}^2 \right) \\
&\leq C \left(h \|\nabla(\mathbf{u} - \Pi_h^k \mathbf{u})\|_{0,\Omega}^2 + h^3 \|\nabla^2(\mathbf{u} - \Pi_h^k \mathbf{u})\|_{0,\Omega}^2 \right) \\
&\leq C h^{2r_{\mathbf{u}}-1} \|\mathbf{u}\|_{r_{\mathbf{u}},\Omega}^2.
\end{aligned}
$$

Obviously, the pressure jump term is treated using the same argument, which completes the proof of (31).

To prove (32) we simply note that by trace inequalities and the stability of the L^2-projection there holds

$$
\begin{aligned}
\|(\nu h)^{\frac{1}{2}} \nabla(\mathbf{u} - \Pi_h^k \mathbf{u})\|_{0,\partial\Omega}^2 &\leq C \|\nu^{\frac{1}{2}} \nabla(\mathbf{u} - \mathcal{I}_h^k \mathbf{u})\|_{0,\Omega}^2 \\
&\quad + C h^2 \sum_{K\in\mathcal{T}_h} \|\nu^{\frac{1}{2}} (\mathbf{u} - \mathcal{I}_h^k \mathbf{u})\|_{2,K}^2 \\
&\leq C \nu h^{2r_{\mathbf{u}}-2} \|\mathbf{u}\|_{r_{\mathbf{u}},\Omega}^2.
\end{aligned}
$$

To conclude, we apply the inequality (33) to the term $\|p - \Pi_h^k p\|_{0,\partial\Omega}^2$. $\square$

Finally, we shall also make use of the following projection operator, based on a Stokes-like problem. For each $\mathbf{u} \in [H^{\frac{3}{2}+\epsilon}(\Omega) \cap H_0^1(\Omega)]^d \cap H_0(\mathrm{div};\Omega)$, we denote by $S_h^k \mathbf{u} \stackrel{\mathrm{def}}{=} (P_h^k \mathbf{u}, R_h^k \mathbf{u}) \in W_h^k$ the unique solution of

$$
\left\{
\begin{aligned}
&(P_h^k \mathbf{u}, \mathbf{v}_h) + a_h(P_h^k \mathbf{u}, \mathbf{v}_h) + b_h(R_h^k \mathbf{u}, \mathbf{v}_h) \\
&\qquad + \gamma j(P_h^k \mathbf{u}, \mathbf{v}_h) = (\mathbf{u}, \mathbf{v}_h) + a_h(\mathbf{u}, \mathbf{v}_h), \\
&- b_h(q_h, P_h^k \mathbf{u}) + j(R_h^k \mathbf{u}, q_h) = 0,
\end{aligned}
\right.
\tag{34}
$$

for all $(\mathbf{v}_h, q_h) \in W_h^k$.

By assuming that $\mathbf{u}$ is also sufficiently regular in time, so that the projection makes sense at each time t, we have the following approximation result, whose proof is based on the results reported in [12].

Lemma 4.4 *Let $\mathbf{u} \in [L^2(0,T; H^r(\Omega) \cap H_0^1(\Omega)]^d \cap H_0(\mathrm{div};\Omega)$. The following error estimate for the projection P_h^k holds:*

$$
\|\mathbf{u} - P_h^k \mathbf{u}\|_{L^\infty(0,T;L^2(\Omega))} \leq C(\nu^{\frac{1}{2}} + h^{\frac{1}{2}}) h^{r_{\mathbf{u}}-1} \|\mathbf{u}\|_{L^2(0,T;H^{r_{\mathbf{u}}}(\Omega))},
\tag{35}
$$

with $C > 0$ independent of ν and h.

By deriving (34) with respect to time, we obtain the similar result for the time derivative of the projection.

Corollary 4.2 *Let* $\mathbf{u} \in [L^2(0,T;H^r(\Omega)\cap H_0^1(\Omega)\cap H_0(\mathrm{div};\Omega))]^d$ *with* $\partial_t \mathbf{u} \in [L^2(0,T;H^{r_\mathbf{u}}(\Omega))]^d$. *Then the following error estimate holds:*

$$\|\partial_t(\mathbf{u} - P_h^k \mathbf{u})\|_{L^\infty(0,T;L^2(\Omega))} \leq C(\nu^{\frac{1}{2}} + h^{\frac{1}{2}}) h^{r_\mathbf{u}-1} \|\partial_t \mathbf{u}\|_{L^2(0,T;H^{r_\mathbf{u}}(\Omega))}. \tag{36}$$

5 Stability

In this section we investigate the wellposedness and some stability properties of the discrete scheme (9).

5.1 Existence and uniqueness of discrete solution

In the sequel, we shall make use of the following discrete pressure and velocity subspaces:

$$\begin{aligned} C_{h,k}^1 &\stackrel{\text{def}}{=} \left\{ q_h \in Q_h^k \,:\, j(q_h, q_h) = 0 \right\}, \\ V_{h,k}^{\text{div}} &\stackrel{\text{def}}{=} \left\{ \mathbf{v}_h \in [V_h^k]^d \,:\, b_h(q_h, \mathbf{v}_h) = 0, \quad \forall q_h \in C_{h,k}^1 \right\}. \end{aligned}$$

In addition, $Q_h^k \backslash C_{h,k}^1$ will stand for the supplementary of $C_{h,k}^1$ in Q_h^k, i.e.,

$$Q_h^k = (Q_h^k \backslash C_{h,k}^1) \oplus C_{h,k}^1.$$

The following lemma ensures, in particular, that $V_{h,k}^{\text{div}}$ is not trivial (i.e. $V_{h,k}^{\text{div}} \neq \{0\}$).

Lemma 5.1 *There exists a constant* $\beta > 0$, *independent of* h, *such that*

$$\inf_{q_h \in C_{h,k}^1} \sup_{\mathbf{v}_h \in [V_h]^d} \frac{|b_h(q_h, \mathbf{v}_h)|}{\|q_h\|_{0,\Omega} \|\mathbf{v}_h\|_{1,\Omega}} \geq \beta.$$

Proof. Let $q_h \in C_{h,k}^1$. From [20, Corollary 2.4], there exists $\mathbf{v}_q \in [H_0^1(\Omega)]^d$ such that

$$\nabla \cdot \mathbf{v}_q = q_h, \quad \|\mathbf{v}_q\|_{1,\Omega} \leq C \|q_h\|_{0,\Omega}. \tag{37}$$

Thus, using integration by parts and (13), we have

$$\begin{aligned} \|q_h\|_{0,\Omega}^2 &= (q_h, \nabla \cdot \mathbf{v}_q) \\ &= (q_h, \nabla \cdot \mathbf{v}_q - \nabla \cdot \Pi_h^k \mathbf{v}_q) + (q_h, \nabla \cdot \Pi_h^k \mathbf{v}_q) \\ &= (\nabla q_h, \mathbf{v}_q - \Pi_h^k \mathbf{v}_q) - \langle q_h, (\Pi_h^k \mathbf{v}_q) \cdot \mathbf{n} \rangle_{\partial\Omega} \\ &\quad + (q_h, \nabla \cdot \Pi_h^k \mathbf{v}_q) \\ &= (\nabla q_h, \mathbf{v}_q - \Pi_h^k \mathbf{v}_q) - b_h(q_h, \Pi_h^k \mathbf{v}_q). \end{aligned} \tag{38}$$

In particular, using the orthogonality of the L^2-projection, Cauchy-Schwartz and Lemma 4.2, and since $j(q_h, q_h) = 0$, we get

$$\begin{aligned}|(\nabla q_h, \mathbf{v}_q - \Pi_h^k \mathbf{v}_q)| &= |(\nabla q_h - \Pi_h^k(\nabla q_h), \mathbf{v}_q - \Pi_h^k \mathbf{v}_q)| \\ &\leq \|\nabla q_h - \Pi_h^k(\nabla q_h)\|_{0,\Omega} \|\mathbf{v}_q - \Pi_h^k \mathbf{v}_q\|_{0,\Omega} \\ &\leq \gamma_3 h^{-\frac{1}{2}} j(q_h, q_h) \|\mathbf{v}_q - \Pi_h^k \mathbf{v}_q\|_{0,\Omega} \\ &= 0.\end{aligned}$$

Thus, from (38), if follows that

$$|b_h(q_h, \Pi_h^k \mathbf{v}_q)| = \|q_h\|_{0,\Omega}^2.$$

In addition, from (18) and (37), we have

$$\begin{aligned}\|\Pi_h^k \mathbf{v}_q\|_{1,\Omega} &\leq C\|\mathbf{v}_q\|_{1,\Omega} \\ &\leq C\|q_h\|_{0,\Omega},\end{aligned}$$

which completes the proof. $\square$

We now state the main result of this paragraph.

Theorem 5.1 *The discrete problem* (9) *with* $\mathbf{u}_{0,h} \in V_{h,k}^{\text{div}}$ *has a unique solution.*

Proof. Problem (9) can be written, in operator form, as

$$\begin{aligned}M\partial_t \mathbf{u}_h + A(\mathbf{u}_h)\mathbf{u}_h + B^T p_h &= M\mathbf{f}, \quad \text{in} \quad \left([V_h^k]^d\right)', \\ B\mathbf{u}_h &= Jp_h, \quad \text{in} \quad [Q_h^k]', \\ \mathbf{u}_h(0) &= \mathbf{u}_{0,h},\end{aligned} \tag{39}$$

with $M \in \mathcal{L}\left([V_h^k]^d, \left([V_h^k]^d\right)'\right)$, $A \in \mathcal{L}\left([V_h^k]^d \times [V_h^k]^d, \left([V_h^k]^d\right)'\right)$, $B \in \mathcal{L}\left([V_h^k]^d, (Q_h^k)'\right)$ and $J \in \mathcal{L}\left(Q_h^k, (Q_h^k)'\right)$ defined by

$$\begin{aligned}\langle M\mathbf{u}_h, \mathbf{v}_h\rangle &\stackrel{\text{def}}{=} (\mathbf{u}_h, \mathbf{v}_h), \\ \langle A(\mathbf{w}_h)\mathbf{u}_h, \mathbf{v}_h\rangle &\stackrel{\text{def}}{=} a_h(\mathbf{u}_h, \mathbf{v}_h) + c_h(\mathbf{w}_h; \mathbf{u}_h, \mathbf{v}_h) + j_{\mathbf{w}_h}(\mathbf{u}_h, \mathbf{v}_h) \\ &\quad + \gamma j(\mathbf{u}_h, \mathbf{v}_h), \\ \langle B\mathbf{v}_h, q_h\rangle &\stackrel{\text{def}}{=} b(q_h, \mathbf{v}_h), \\ \langle Jp_h, q_h\rangle &\stackrel{\text{def}}{=} j(p_h, q_h).\end{aligned}$$

We also introduce the operator $B^1 \in \mathcal{L}\left([V_h^k]^d, (C_{h,k}^1)'\right)$ defined by

$$\langle B^1\mathbf{v}_h, q_h\rangle \stackrel{\text{def}}{=} b_h(q_n, \mathbf{v}_h), \quad \forall(\mathbf{v}_h, q_h) \in [V_h^k]^d \times C_{h,k}^1,$$

in other words,

$$B^1\mathbf{v}_h \stackrel{\text{def}}{=} (B\mathbf{v}_h)_{|C^1_{h,k}}, \quad \forall \mathbf{v}_h \in [V_h^k]^d.$$

From Lemma 5.1, it follows that B^1 is surjective and $(B^1)^{\mathrm{T}}$ is injective (see [20, page 58]). We then deduce that $V_{h,k}^{\mathrm{div}} \stackrel{\text{def}}{=} \mathrm{Ker}(B^1) \neq \{0\}$.

Let us consider the following reduced formulation (derived from (9) with $(\mathbf{v}_h, q_h) \in V_{h,k}^{\mathrm{div}} \times (Q_h^k \backslash C^1_{h,k})$): find $(\mathbf{u}_h(t), \tilde{p}_h(t)) \in V_{h,k}^{\mathrm{div}} \times (Q_h^k \backslash C^1_{h,k})$ such that

$$\begin{aligned} M\partial_t \mathbf{u}_h + A(\mathbf{u}_h)\mathbf{u}_h + B^T\tilde{p}_h &= M\mathbf{f}, \quad \text{in} \quad \left(V_{h,k}^{\mathrm{div}}\right)', \\ B\mathbf{u}_h &= J\tilde{p}_h, \quad \text{in} \quad \left(Q_h^k \backslash C^1_{h,k}\right)', \\ \mathbf{u}_h(0) &= \mathbf{u}_{0,h}. \end{aligned} \tag{40}$$

Since, by construction, $C^1_{h,k} = \mathrm{Ker}(J)$, we conclude that J is invertible in $Q_h^k \backslash C^1_{h,k}$. Hence, from (40), we have

$$\tilde{p}_h = J^{-1}_{|Q_h^k \backslash C^1_{h,k}} B\mathbf{u}_h. \tag{41}$$

By plugging this expression into the first equation of (40), we obtain that $\mathbf{u}_h(t) \in V_{h,k}^{\mathrm{div}}$ solves

$$\begin{aligned} M\partial_t \mathbf{u}_h + A(\mathbf{u}_h)\mathbf{u}_h + B^T J^{-1}_{|Q_h^k \backslash C^1_{h,k}} B\mathbf{u}_h &= M\mathbf{f}, \quad \text{in} \quad \left(V_{h,k}^{\mathrm{div}}\right)', \\ \mathbf{u}_h(0) &= \mathbf{u}_{0,h}, \end{aligned}$$

which is a standard Cauchy problem for $\mathbf{u}_h$. Existence and uniqueness of $\mathbf{u}_h$ follows by the Lipschitz continuity of A. We may then recover $\tilde{p}_h$ uniquely from (41). Therefore, the reduced problem (40) has a unique solution. On the other hand, from the first equation of (40), if follows that

$$M\partial_t \mathbf{u}_h + A(\mathbf{u}_h)\mathbf{u}_h + B^T\tilde{p}_h - M\mathbf{f} \in \left(\mathrm{Ker}(B^1)\right)^0,$$

with $\left(\mathrm{Ker}(B^1)\right)^0$ standing for the polar set of $\mathrm{Ker}(B^1)$. From Lemma 5.1, it follows that B^1 is an isomorphism from $C^1_{h,k}$ onto $\left(\mathrm{Ker}(B^1)\right)^0$ (see [20, page 58]). Thus, there exists a unique $p^1 \in C^1_{h,k}$ such that

$$M\partial_t \mathbf{u}_h + A(\mathbf{u}_h)\mathbf{u}_h + B^T\tilde{p}_h - M\mathbf{f} = (B^1)^{\mathrm{T}} p^1 \quad \text{in} \quad \left([V_h^k]^d\right)'. \tag{42}$$

Therefore, from (42) and (40), and by noticing that $(B^1)^{\mathrm{T}}p^1 = B^{\mathrm{T}}p^1$ and $Jp^1 = 0$, if follows that problem (9) has a unique solution, given by $(\mathbf{u}_h, p_h \stackrel{\text{def}}{=} \tilde{p}_h - p^1)$. $\square$

Remark 5.1 *In order to ensure convergence, in the following we shall set* $\mathbf{u}_{0,h} = P_h^k \mathbf{u}_0 \in V_{h,k}^{\mathrm{div}}$.

5.2 Coercivity

The following Lemma provides control (uniform in ν) of the divergence constraint through the stabilization terms.

Lemma 5.2 (Divergence control) *Assume* $(\mathbf{u}_h, p_h) \in W_h^k$ *be a solution of* (9). *There exists a constant* $C > 0$*, depending only on the mesh geometry, such that*

$$C\|h^{\frac{1}{2}}\nabla\cdot\mathbf{u}_h\|_{0,\Omega}^2 \leq \mathbf{J}\big[\mathbf{0};(\mathbf{u}_h,p_h),(\mathbf{u}_h,p_h)\big] + \|\mathbf{u}_h\cdot\mathbf{n}\|_{0,\partial\Omega}^2.$$

Proof. By testing (9) with $\mathbf{v}_h = \mathbf{0}$ we get

$$(q_h, \nabla\cdot\mathbf{u}_h) - \langle q_h, \mathbf{u}_h\cdot\mathbf{n}\rangle_{\partial\Omega} + j(p_h,q_h) = 0.$$

Thus, taking $q_h = \pi_h^*(h\nabla\cdot\mathbf{u}_h)$ yields

$$\begin{aligned}\|h^{\frac{1}{2}}\nabla\cdot\mathbf{u}_h\|_{0,\Omega}^2 &+ (\nabla\cdot\mathbf{u}_h, \pi_h^*(h\nabla\cdot\mathbf{u}_h) - h\nabla\cdot\mathbf{u}_h)\\ &= \langle\pi_h^*(h\nabla\cdot\mathbf{u}_h), \mathbf{u}_h\cdot\mathbf{n}\rangle_{\partial\Omega} - j(p_h, \pi_h^*(h\nabla\cdot\mathbf{u}_h)).\end{aligned}$$

It follows then, by the quasi-uniformity of the mesh, a Cauchy-Schwarz inequality, a trace inequality and an inverse inequality, that

$$\begin{aligned}\frac{1}{2}\|h^{\frac{1}{2}}\nabla\cdot\mathbf{u}_h\|_{0,\Omega}^2 \leq{}& \frac{1}{2}\|h^{\frac{1}{2}}\big(\pi_h^*(\nabla\cdot\mathbf{u}_h) - \nabla\cdot\mathbf{u}_h\big)\|_{0,\Omega}^2\\ &+ C\Big(\|\mathbf{u}_h\cdot\mathbf{n}\|_{0,\partial\Omega} + j^{\frac{1}{2}}(p_h,p_h)\Big)\|h^{\frac{1}{2}}\pi_h^*(\nabla\cdot\mathbf{u}_h)\|_{0,\Omega}.\end{aligned}$$

Therefore, using the triangle inequality, this yields

$$\begin{aligned}\frac{1}{2}\|h^{\frac{1}{2}}\nabla\cdot\mathbf{u}_h\|_{0,\Omega}^2 \leq{}& C\|h^{\frac{1}{2}}\big(\pi_h^*(\nabla\cdot\mathbf{u}_h) - \nabla\cdot\mathbf{u}_h\big)\|_{0,\Omega}^2\\ &+ C\left(\|\mathbf{u}_h\cdot\mathbf{n}\|_{0,\partial\Omega}^2 + j(p_h,p_h)\right)\\ &+ C\Big(\|\mathbf{u}_h\cdot\mathbf{n}\|_{0,\partial\Omega} + j^{\frac{1}{2}}(p_h,p_h)\Big)\|h^{\frac{1}{2}}\nabla\cdot\mathbf{u}_h\|_{0,\Omega}.\end{aligned}$$

We conclude using a Young's inequality and the interpolation result (27). $\square$

Using Lemma 5.2 we may now show that the bilinear form is coercive for the triple norm $|\!|\!|\cdot|\!|\!|_{\mathbf{w}_h}$.

Lemma 5.3 (Coercivity) *There exits a constant* $C_{\rm A} > 0$*, depending only on* Ω *and* γ_ν*, such that*

$$(\mathbf{A}+\mathbf{J})\big[\mathbf{w}_h;(\mathbf{v}_h,q_h),(\mathbf{v}_h,q_h)\big] \geq C_{\rm A}|\!|\!|(\mathbf{v}_h,q_h)|\!|\!|_{\mathbf{w}_h}^2,$$

for all $(\mathbf{w}_h,(\mathbf{v}_h,q_h)) \in [V_h^k]^d \times W_h^k$.

Proof. From (9) we have

$$
\begin{aligned}
(\mathbf{A}+\mathbf{J})\big[\mathbf{w}_h;(\mathbf{v}_h,q_h),(\mathbf{v}_h,q_h)\big] &\geq 2\|\nu^{\frac{1}{2}}\boldsymbol{\varepsilon}(\mathbf{v}_h)\|_{0,\Omega}^2 \\
&+\mathbf{J}\big[\mathbf{w}_h;(\mathbf{v}_h,q_h),(\mathbf{v}_h,q_h)\big]+\|\gamma_\nu^{\frac{1}{2}}\,(\nu/h)^{\frac{1}{2}}\,\mathbf{v}_h\|_{0,\partial\Omega}^2 \\
&+\|\mathbf{v}_h\cdot\mathbf{n}\|_{0,\partial\Omega}^2-\langle 4\nu\boldsymbol{\varepsilon}(\mathbf{v}_h)\mathbf{n},\mathbf{v}_h\rangle_{\partial\Omega}\,,
\end{aligned}
\tag{43}
$$

where we used the fact that, after integration by parts,

$$
(\mathbf{w}_h\cdot\nabla\mathbf{v}_h,\mathbf{v}_h)=\frac{1}{2}\big[\,\langle\mathbf{w}_h\cdot\mathbf{n}\mathbf{v}_h,\mathbf{v}_h\rangle_{\partial\Omega}-(\nabla\cdot\mathbf{w}_h\mathbf{v}_h,\mathbf{v}_h)\big].
$$

The last term in (43) can be bounded using the Cauchy-Schwarz inequality followed by (25) and the quasi-uniformity of the mesh (7), to obtain

$$
|\,\langle 4\nu\boldsymbol{\varepsilon}(\mathbf{v}_h)\mathbf{n},\mathbf{v}_h\rangle_{\partial\Omega}\,|\leq 8\frac{C_{\mathrm{T}}}{C_{\mathrm{U}}\gamma_\nu}\|\nu^{\frac{1}{2}}\boldsymbol{\varepsilon}(\mathbf{v}_h)\|_{0,\Omega}^2+\frac{1}{2}\|\gamma_\nu^{\frac{1}{2}}\left(\frac{\nu}{h}\right)^{\frac{1}{2}}\mathbf{v}_h\|_{0,\partial\Omega}^2.
$$

In the sequel we will assume that

$$
\gamma_\nu>4\frac{C_{\mathrm{T}}}{C_{\mathrm{U}}}>0,
\tag{44}
$$

and therefore

$$
\lambda(\gamma_\nu)\stackrel{\mathrm{def}}{=}2-8\frac{C_{\mathrm{T}}}{C_{\mathrm{U}}\gamma_\nu}>0.
$$

From (43), we then get

$$
\begin{aligned}
(\mathbf{A}+\mathbf{J})\big[\mathbf{w}_h;(\mathbf{v}_h,q_h),(\mathbf{v}_h,q_h)\big] &\geq \lambda(\gamma_\nu)\|\nu^{\frac{1}{2}}\boldsymbol{\varepsilon}(\mathbf{v}_h)\|_{0,\Omega}^2 \\
&+\mathbf{J}\big[\mathbf{w}_h;(\mathbf{v}_h,q_h),(\mathbf{v}_h,q_h)\big]+\frac{1}{2}\|\gamma_\nu^{\frac{1}{2}}\,(\nu/h)^{\frac{1}{2}}\,\mathbf{v}_h\|_{0,\partial\Omega}^2 \\
&+\|\mathbf{v}_h\cdot\mathbf{n}\|_{0,\partial\Omega}^2,
\end{aligned}
$$

and consequently

$$
\begin{aligned}
&(\mathbf{A}+\mathbf{J})\big[\mathbf{w}_h;(\mathbf{v}_h,q_h),(\mathbf{v}_h,q_h)\big] \\
&\geq \min\left\{\lambda(\gamma_\nu),\frac{\gamma_\nu}{4h}\right\}\left(\|\nu^{\frac{1}{2}}\boldsymbol{\varepsilon}(\mathbf{v}_h)\|_{0,\Omega}^2+\|\nu^{\frac{1}{2}}\mathbf{v}_h\|_{0,\partial\Omega}^2\right) \\
&\quad+\mathbf{J}\big[\mathbf{w}_h;(\mathbf{v}_h,q_h),(\mathbf{v}_h,q_h)\big]+\frac{1}{4}\|\gamma_\nu^{\frac{1}{2}}\,(\nu/h)^{\frac{1}{2}}\,\mathbf{v}_h\|_{0,\partial\Omega}^2 \\
&\quad+\|\mathbf{v}_h\cdot\mathbf{n}\|_{0,\partial\Omega}^2.
\end{aligned}
$$

In particular, by choosing (accordingly with (44))

$$
\gamma_\nu\stackrel{\mathrm{def}}{=}\frac{1}{8}+4\frac{C_{\mathrm{T}}}{C_{\mathrm{U}}},
$$

and since $0<h\leq 1$, one obtains

$$
\lambda(\gamma_\nu)<\frac{\gamma_\nu}{4h}.
$$

We conclude the proof using Korn's inequality (see [7]) and Lemma 5.2. $\square$

6 Convergence

We now prove convergence first of the velocities and then of the pressures. Since the problem decomposes into one linear part and one non-linear part it is convenient first to recall a preliminary result regarding the continuity of the Stokes system from [10].

Lemma 6.1 *There exists a constant $C > 0$, independent of ν and h, such that*

$$a_h(\mathbf{v}, \mathbf{v}_h) - b_h(q, \mathbf{v}_h) + b_h(q_h, \mathbf{v}) \leq C|\!|\!|(\mathbf{v}, q)|\!|\!|\,|\!|\!|(\mathbf{v}_h, q_h)|\!|\!|_{\mathbf{0}},$$

for all $(q, \mathbf{v}) \in \left[(V_h^k)^{\perp} \times ([V_h^k]^d)^{\perp}\right] \cap [H^2(\mathcal{T}_h)]^{d+1}$ *and* $(q_h, \mathbf{v}_h) \in V_h^k \times [V_h^k]^d$.

Proof. Using Cauchy-Schwarz and the trace inequality (25) and since $0 < h, \nu \leq 1$, for the first term one readily obtains

$$a_h(\mathbf{v}, \mathbf{v}_h) \leq C|\!|\!|(\mathbf{v}, 0)|\!|\!|\,|\!|\!|(\mathbf{v}_h, 0)|\!|\!|_{\mathbf{0}}.$$

For the second term we have, using the orthogonality of q (to V_h^k) and the interpolation estimate (27),

$$\begin{aligned} b_h(q, \mathbf{v}_h) &= -(q, \nabla \cdot \mathbf{w}_h - \pi_h^*(\nabla \cdot \mathbf{v}_h)) + \langle q, \mathbf{v}_h \cdot \mathbf{n} \rangle_{\partial\Omega} \\ &\leq \|h^{-\frac{1}{2}} q\|_{0,\Omega} \|h^{\frac{1}{2}}(\nabla \cdot \mathbf{v}_h - \pi_h^*(\nabla \cdot \mathbf{v}_h))\|_{0,\Omega} \\ &\quad + \|q\|_{0,\partial\Omega} \|\mathbf{v}_h \cdot \mathbf{n}\|_{0,\partial\Omega} \\ &\leq C|\!|\!|(0, q)|\!|\!|\,|\!|\!|(\mathbf{w}_h, 0)|\!|\!|_{\mathbf{0}}. \end{aligned}$$

In a similar fashion, after integration by parts in the third term, one obtains

$$\begin{aligned} b_h(q_h, \mathbf{v}) &= -(q_h, \nabla \cdot \mathbf{v}) + \langle q_h, \mathbf{v} \cdot \mathbf{n} \rangle_{\partial\Omega} \\ &= (\nabla q_h, \mathbf{v}) \\ &= (\nabla q_h - \pi_h^*(\nabla q_h), \mathbf{v}) \\ &\leq \|h^{\frac{1}{2}}(\nabla q_h - \pi_h^*(\nabla q_h))\|_{0,\Omega} \|h^{-\frac{1}{2}} \mathbf{v}\|_{0,\Omega} \\ &\leq C|\!|\!|(\mathbf{v}, 0)|\!|\!|\,|\!|\!|(\mathbf{0}, q_h)|\!|\!|_{\mathbf{0}}. \end{aligned}$$

Hence, the proof is complete. □

6.1 Velocity energy norm error estimate

The following theorem states the main result of this paragraph.

Theorem 6.1 *Let $(\mathbf{u}, p)$ the solution of (1), $(\mathbf{u}_h, p_h) \in W_h^k$ the solution of (9), with $\mathbf{u}_{0,h} = P_h^k \mathbf{u}_0$, and assume that $(\mathbf{u}, p)$ has the minimal regularity (4). Then, the following optimal approximation estimates hold*

$$\begin{aligned} \|\Pi_h^k \mathbf{u} - \mathbf{u}_h\|^2_{L^\infty(0,T;L^2(\Omega))} &\leq \|\Pi_h^k \mathbf{u}_0 - \mathbf{u}_{0,h}\|^2_{0,\Omega} \\ &+ c_{\exp} \int_0^T \Big(c_1 |\!|\!|(\mathbf{u} - \Pi_h^k \mathbf{u}, 0)|\!|\!|^2 + c_2 \mathbf{J}\left[\mathbf{0}; (\Pi_h^k \mathbf{u}, \Pi_h^k p), (\Pi_h^k \mathbf{u}, \Pi_h^k p)\right] \Big)\, \mathrm{d}t, \quad (45) \end{aligned}$$

and

$$\int_0^T |||(\Pi_h^k \mathbf{u} - \mathbf{u}_h, \Pi_h^k p - p_h)|||_{\mathbf{u}_h}^2 \, \mathrm{d}t \le \|\Pi_h^k \mathbf{u}_0 - \mathbf{u}_{0,h}\|_{0,\Omega}^2 + c_{\exp} \int_0^T \left(c_1 |||(\mathbf{u} - \Pi_h^k \mathbf{u}, 0)|||^2 + c_2 \mathbf{J}\left[\mathbf{0}; (\Pi_h^k \mathbf{u}, \Pi_h^k p), (\Pi_h^k \mathbf{u}, \Pi_h^k p)\right] \right) \mathrm{d}t, \quad (46)$$

with

$$\begin{aligned} c_{\exp} &\stackrel{\text{def}}{=} e^{CT\left(h\|\mathbf{u}\|^2_{L^\infty(0,T;W^{1,\infty}(\Omega))} + \|\mathbf{u}\|_{L^\infty(0,T;W^{1,\infty}(\Omega))}\right)}, \\ c_1 &\stackrel{\text{def}}{=} C\left(1 + h^{\frac{1}{2}} \|\mathbf{u}\|_{L^\infty(0,T;W^{1,\infty}(\Omega))}\right), \\ c_2 &\stackrel{\text{def}}{=} C\left(1 + \|\mathbf{u}\|^2_{L^\infty(0,T;L^\infty(\Omega))} + h^2 \|\mathbf{u}\|^2_{L^\infty(0,T;W^{1,\infty}(\Omega))}\right), \end{aligned}$$

with $C > 0$ a positive constant independent of ν and h.

We stress that the constants in the above theorem have no explicit dependence on ν. Before proving the main convergence theorem we state two immediate consequences in the form of corollaries.

Corollary 6.1 *Under the hypothesis of the previous theorem, assuming that the exact solution $(\mathbf{u}, p)$ has the regularity given in (5) and that $\nu < h$, the following error estimates hold*

$$\|\mathbf{u} - \mathbf{u}_h\|^2_{L^\infty(0,T;L^2(\Omega))} \le C(c_{\exp}, c_1, c_2) \Big(h^{2r_{\mathbf{u}}-1} \|\mathbf{u}\|^2_{L^2(0,T;H^{r_{\mathbf{u}}}(\Omega))} + h^{2r_p - 1} \|p\|^2_{L^2(0,T;H^{r_p}(\Omega))} \Big),$$

and

$$\int_0^T |||(\mathbf{u} - \mathbf{u}_h, p - p_h)|||^2_{\mathbf{u}_h} \, \mathrm{d}t \le C(c_{\exp}, c_1, c_2) \Big(h^{2r_{\mathbf{u}}-1} \|\mathbf{u}\|^2_{L^2(0,T;H^{r_{\mathbf{u}}}(\Omega))} + h^{2r_p - 1} \|p\|^2_{L^2(0,T;H^{r_p}(\Omega))} \Big),$$

with $r_{\mathbf{u}} \stackrel{\text{def}}{=} \min(r, k+1)$, $r_p \stackrel{\text{def}}{=} \min(s, k+1)$ and $C(c_{\exp}, c_1, c_2) > 0$ a positive constant independent of ν and h.

Proof. Immediate by a triangle inequality, the result of Theorem 6.1 and approximation (Lemmas 4.3 and 4.4). □

Corollary 6.2 *Under the hypothesis of the previous corollary, the following error estimate holds:*

$$\|\mathbf{u}-\mathbf{u}_h\|^2_{L^\infty(0,T;L^\infty(\Omega))} \leq Ch^2\|\mathbf{u}\|^2_{L^\infty(0,T;W^{1,\infty}(\Omega))} + C(c_{\exp},c_1,c_2)h^{-d}\left(h^{2r_{\mathbf{u}}-1}\|\mathbf{u}\|^2_{L^2(0,T;H^{r_{\mathbf{u}}}(\Omega))} + h^{2r_p-1}\|p\|^2_{L^2(0,T;H^r_p(\Omega))}\right),$$

with $C>0$ a positive constant independent of ν and h. In particular, there holds $\mathbf{u}_h \in L^\infty(0,T;L^\infty(\Omega))$.

Proof. Immediate using approximation (22), an inverse inequality and (45). □

6.2 Proof of Theorem 6.1

In the following, $\epsilon_i > 0$ for $i = 1,2,\ldots$, represents a free positive constant to be fixed later on. We denote the discrete and projection errors as

$$\begin{aligned} \boldsymbol{\theta}_h &\stackrel{\text{def}}{=} \Pi^k_h\mathbf{u} - \mathbf{u}_h, \quad \boldsymbol{\theta}^\pi \stackrel{\text{def}}{=} \mathbf{u} - \Pi^k_h\mathbf{u},\\ y_h &\stackrel{\text{def}}{=} \Pi^k_h p - p_h, \quad y^\pi = p - \Pi^k_h p, \end{aligned} \tag{47}$$

which gives

$$\boldsymbol{\theta}_h = \mathbf{u} - \mathbf{u}_h - \boldsymbol{\theta}^\pi, \quad y_h = p - p_h - y^\pi. \tag{48}$$

Note that, since $\mathbf{u} \in H^1(0,T;L^2(\Omega))$, we may deduce that $\boldsymbol{\theta}_h \in H^1(0,T;L^2(\Omega))$. Using coercivity (Lemma 5.3) we then get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{\theta}_h\|^2_{0,2} + C_{\mathrm{A}}|||(\boldsymbol{\theta}_h,y_h)|||^2_{\mathbf{u}_h} \leq (\partial_t\boldsymbol{\theta}_h,\boldsymbol{\theta}_h) + (\mathbf{A}+\mathbf{J})\big[\mathbf{u}_h;(\boldsymbol{\theta}_h,y_h),(\boldsymbol{\theta}_h,y_h)\big].$$

Hence, from (47)-(48) and the tri-linearity of $\mathbf{A}$ and $\mathbf{J}$, we have

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{\theta}_h\|^2_{0,2} + C_{\mathrm{A}}|||(\boldsymbol{\theta}_h,y_h)|||^2_{\mathbf{u}_h} \leq{}& (\partial_t(\mathbf{u}-\mathbf{u}_h),\boldsymbol{\theta}_h) - (\partial_t\boldsymbol{\theta}^\pi,\boldsymbol{\theta}_h)\\ &+ (\mathbf{A}+\mathbf{J})\big[\mathbf{u}_h;(\Pi^k_h\mathbf{u},\Pi^k_h p)(\boldsymbol{\theta}_h,y_h)\big]\\ &- (\mathbf{A}+\mathbf{J})\big[\mathbf{u}_h;(\mathbf{u}_h,p_h),(\boldsymbol{\theta}_h,y_h)\big].\end{aligned}$$

Thus, by testing the approximate Galerkin orthogonality (Lemma 3.1) with $(\mathbf{v}_h,q_h) = (\boldsymbol{\theta}_h,y_h)$ and since since $(\partial_t\boldsymbol{\theta}^\pi,\boldsymbol{\theta}_h) = 0$, we obtain

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{\theta}_h\|^2_{0,2} + C_{\mathrm{A}}|||(\boldsymbol{\theta}_h,y_h)|||^2_{\mathbf{u}_h} \leq{}& (\mathbf{A}+\mathbf{J})\big[\mathbf{u}_h;(\Pi^k_h\mathbf{u},\Pi^k_h p)(\boldsymbol{\theta}_h,y_h)\big]\\ &- \mathbf{A}\big[\mathbf{u};(\mathbf{u},p),(\boldsymbol{\theta}_h,y_h)\big].\end{aligned}$$

Writing out all the terms of $\mathbf{A}$ and $\mathbf{J}$, as given in (10), we get

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{\theta}_h\|_{0,2}^2 + C_{\mathrm{A}}|||(\boldsymbol{\theta}_h, y_h)|||_{\mathbf{u}_h}^2 &\leq -a_h(\boldsymbol{\theta}^\pi, \boldsymbol{\theta}_h) - b_h(y^\pi, \boldsymbol{\theta}_h) + b_h(y_h, \boldsymbol{\theta}^\pi)\\ &+ \gamma j(\Pi_h^k \mathbf{u}, \boldsymbol{\theta}_h) + j(\Pi_h^k p, y_h) + c_h(\mathbf{u}_h; \Pi_h^k \mathbf{u}, \boldsymbol{\theta}_h) - c(\mathbf{u}; \mathbf{u}, \boldsymbol{\theta}_h)\\ &+ j_{\mathbf{u}_h}(\Pi_h^k \mathbf{u}, \boldsymbol{\theta}_h).\end{aligned}$$

Now we may use the continuity of the Stokes system (Lemma 6.1) to obtain,

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{\theta}_h\|_{0,2}^2 + C_{\mathrm{A}}|||(\boldsymbol{\theta}_h, y_h)|||_{\mathbf{u}_h}^2\\ &\quad\leq C_1\Big(|||(\boldsymbol{\theta}^\pi, y^\pi)||| + \mathbf{J}^{\frac{1}{2}}\left[\mathbf{0}; (\Pi_h^k \mathbf{u}, \Pi_h^k p), (\Pi_h^k \mathbf{u}, \Pi_h^k p)\right]\Big)|||(\boldsymbol{\theta}_h, y_h)|||_{\mathbf{0}}\\ &\quad+ (\mathbf{u}_h \cdot \nabla \Pi_h^k \mathbf{u}, \boldsymbol{\theta}_h) + \frac{1}{2}(\nabla \cdot \mathbf{u}_h \Pi_h^k \mathbf{u}, \boldsymbol{\theta}_h)\\ &\quad\quad- \frac{1}{2}\langle \mathbf{u}_h \cdot \mathbf{n} \Pi_h^k \mathbf{u}, \boldsymbol{\theta}_h\rangle_{\partial\Omega} - (\mathbf{u} \cdot \nabla \mathbf{u}, \boldsymbol{\theta}_h)\\ &\quad\quad+ \sum_{K \in \mathcal{T}_h} \int_{\partial K} h_K^2 |\mathcal{I}_h^1 \mathbf{u}_h|^2 [\![\nabla \Pi_h^k \mathbf{u}]\!] : [\![\nabla \boldsymbol{\theta}_h]\!] \,\mathrm{d}\boldsymbol{s}.\end{aligned}$$

Using (48), this leads to

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{\theta}_h\|_{0,2}^2 + C_{\mathrm{A}}|||(\boldsymbol{\theta}_h, y_h)|||_{\mathbf{u}_h}^2 \leq\\ &\quad C_1\Big(|||(\boldsymbol{\theta}^\pi, y^\pi)||| + \mathbf{J}^{\frac{1}{2}}\left[\mathbf{0}; (\Pi_h^k \mathbf{u}, \Pi_h^k p), (\Pi_h^k \mathbf{u}, \Pi_h^k p)\right]\Big)|||(\boldsymbol{\theta}_h, y_h)|||_{\mathbf{0}}\\ &\quad- (\mathbf{u}_h \cdot \nabla \boldsymbol{\theta}^\pi, \boldsymbol{\theta}_h) - \frac{1}{2}(\nabla \cdot \mathbf{u}_h \boldsymbol{\theta}^\pi, \boldsymbol{\theta}_h)\\ &\quad+ \frac{1}{2}\langle \mathbf{u}_h \cdot \mathbf{n} \boldsymbol{\theta}^\pi, \boldsymbol{\theta}_h\rangle_{\partial\Omega} + ((\mathbf{u}_h - \mathbf{u}) \cdot \nabla \mathbf{u}, \boldsymbol{\theta}_h)\\ &\quad+ \frac{1}{2}(\nabla \cdot \mathbf{u}_h \mathbf{u}, \boldsymbol{\theta}_h) + \sum_{K \in \mathcal{T}_h} \int_{\partial K} h_K^2 |\mathcal{I}_h^1 \mathbf{u}_h \cdot \mathbf{n}|^2 [\![\nabla \Pi_h^k \mathbf{u}]\!] : [\![\nabla \boldsymbol{\theta}_h]\!] \,\mathrm{d}\boldsymbol{s},\end{aligned}$$

which, after integration by parts in the convective term, gives

$$
\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{\theta}_h\|_{0,2}^2 + C_{\mathrm{A}}|\!|\!|(\boldsymbol{\theta}_h, y_h)|\!|\!|_{\mathbf{u}_h}^2 &\leq \frac{C_1\epsilon_1}{2}|\!|\!|(\boldsymbol{\theta}_h, y_h)|\!|\!|_{\mathbf{0}}^2 \\
&+ \frac{C_1}{2\epsilon_1}\Big(|\!|\!|(\boldsymbol{\theta}^\pi, y^\pi)|\!|\!|^2 + \mathbf{J}\big[\mathbf{0}, (\Pi_h^k\mathbf{u}, \Pi_h^k p), (\Pi_h^k\mathbf{u}, \Pi_h^k p)\big]\Big) \\
&+ \underbrace{(\boldsymbol{\theta}^\pi, \mathbf{u}_h\cdot\nabla\boldsymbol{\theta}_h)}_{T_1} + \underbrace{\frac{1}{2}(\nabla\cdot\mathbf{u}_h\boldsymbol{\theta}^\pi, \boldsymbol{\theta}_h)}_{T_2} - \underbrace{\frac{1}{2}\langle\mathbf{u}_h\cdot\mathbf{n}\boldsymbol{\theta}^\pi, \boldsymbol{\theta}_h\rangle_{\partial\Omega}}_{T_3} \\
&+ \underbrace{((\mathbf{u}_h - \mathbf{u})\cdot\nabla\mathbf{u}, \boldsymbol{\theta}_h)}_{T_4} + \underbrace{\frac{1}{2}(\nabla\cdot\mathbf{u}_h\mathbf{u}, \boldsymbol{\theta}_h)}_{T_5} \\
&+ \underbrace{\sum_{K\in\mathcal{T}_h}\int_{\partial K} h_K^2|\mathcal{I}_h^1\mathbf{u}_h\cdot\mathbf{n}|^2[\![\nabla\Pi_h^k\mathbf{u}]\!] : [\![\nabla\boldsymbol{\theta}_h]\!]\,\mathrm{d}\boldsymbol{s}}_{T_6}\,.
\end{aligned} \tag{49}
$$

In the next paragraphs we analyze the terms T_i, $i = 1, 2, \ldots, 6$. Using the orthogonality of the L^2-projection, approximation and Lemma 4.2, we have

$$
\begin{aligned}
T_1 = &\underbrace{(\boldsymbol{\theta}^\pi, (\mathbf{u}_h - \mathcal{I}_h^1\mathbf{u}_h)\cdot\nabla\boldsymbol{\theta}_h)}_{T_{1,1}} \\
&+ \underbrace{(\boldsymbol{\theta}^\pi, \mathcal{I}_h^1\mathbf{u}_h\cdot\nabla\boldsymbol{\theta}_h - \pi_h^*(\mathcal{I}_h^1\mathbf{u}_h\cdot\nabla\boldsymbol{\theta}_h))}_{T_{2,2}}.
\end{aligned}
$$

In the first term, we use the local interpolation property of the $\mathbb{P}_1$-interpolant followed by an inverse inequality showing that

$$
\begin{aligned}
\|\mathbf{u}_h - \mathcal{I}_h^1\mathbf{u}_h\|_{0,K} &\leq C_2 h_K^2|\mathbf{u}_h|_{2,K} \\
&\leq C_3\|\mathbf{u}_h - \mathcal{I}_h^1\mathbf{u}\|_{0,K}.
\end{aligned}
$$

Using this inequality, for the first term of $T_{1,1}$ we have

$$
\begin{aligned}
T_{1,1} &\leq C_4\sum_{K\in\mathcal{T}_h}\|\mathbf{u}_h - \mathcal{I}_h^1\mathbf{u}_h\|_{0,K}\|[\nabla\boldsymbol{\theta}_h]^{\mathrm{T}}\boldsymbol{\theta}^\pi\|_{0,K} \\
&\leq C_5\sum_{K\in\mathcal{T}_h}\|\mathbf{u}_h - \mathcal{I}_h^1\mathbf{u}\|_{0,K}\|[\nabla\boldsymbol{\theta}_h]^{\mathrm{T}}\boldsymbol{\theta}^\pi\|_{0,K}.
\end{aligned}
$$

We now use the decomposition (48), to obtain

$$
T_{1,1} \leq C_5\sum_{K\in\mathcal{T}_h}\left(\|\boldsymbol{\theta}_h\|_{0,K} + \|\Pi_h^k\mathbf{u} - \mathcal{I}_h^1\mathbf{u}\|_{0,K}\right)\|[\nabla\boldsymbol{\theta}_h]^{\mathrm{T}}\boldsymbol{\theta}^\pi\|_{0,K}.
$$

Thus, using inverse inequalities (16) and the L^∞-stability of Π_h^k (22), one gets

$$
\begin{aligned}
T_{1,1} &\leq C_6 h^{-1} \sum_{K\in\mathcal{T}_h} \|\boldsymbol{\theta}^\pi\|_{0,\infty,K} \|\boldsymbol{\theta}_h\|_{0,K}^2 \\
&\quad + C_7 \sum_{K\in\mathcal{T}_h} \|\Pi_h^k \mathbf{u} - \mathcal{I}_h^1 \mathbf{u}\|_{0,\infty,K} \|\nabla \boldsymbol{\theta}_h\|_{0,K} \|\boldsymbol{\theta}^\pi\|_{0,K} \\
&\leq C_8 h^{-1} \|\boldsymbol{\theta}^\pi\|_{0,\infty,\Omega} \|\boldsymbol{\theta}_h\|_{0,\Omega}^2 \\
&\quad + C_9 h^{-\frac{1}{2}} \|\Pi_h^k \mathbf{u} - \mathcal{I}_h^1 \mathbf{u}\|_{0,\infty,\Omega} \sum_{K\in\mathcal{T}_h} \|\boldsymbol{\theta}_h\|_{0,K} \|h^{-\frac{1}{2}} \boldsymbol{\theta}^\pi\|_{0,K} \\
&\leq C_{10} h^{-1} \|\boldsymbol{\theta}^\pi\|_{0,\infty,\Omega} \|\boldsymbol{\theta}_h\|_{0,\Omega}^2 \\
&\quad + C_{11} h^{-\frac{1}{2}} \|\mathbf{u} - \mathcal{I}_h^1 \mathbf{u}\|_{0,\infty,\Omega} \left(\|\boldsymbol{\theta}_h\|_{0,\Omega}^2 + \|h^{-\frac{1}{2}} \boldsymbol{\theta}^\pi\|_{0,\Omega}^2 \right),
\end{aligned}
$$

which, in combination with (20) and approximation, leads to

$$
\begin{aligned}
T_{1,1} &\leq C_{12} \|\nabla \mathbf{u}\|_{0,\infty,\Omega} \left(\|\boldsymbol{\theta}_h\|_{0,\Omega}^2 + h^{\frac{1}{2}} \|h^{-\frac{1}{2}} \boldsymbol{\theta}^\pi\|_{0,\Omega}^2 \right) \\
&\leq C_{13} \|\nabla \mathbf{u}\|_{0,\infty,\Omega} \left(\|\boldsymbol{\theta}_h\|_{0,\Omega}^2 + h^{\frac{1}{2}} |\!|\!| (\boldsymbol{\theta}^\pi, 0) |\!|\!|^2 \right).
\end{aligned}
$$

Finally, using Cauchy-Schwarz and (4.2), we obtain

$$
\begin{aligned}
T_{1,2} &\leq \|h^{-\frac{1}{2}} \boldsymbol{\theta}^\pi\|_{0,\Omega} \|h^{\frac{1}{2}} \left(\mathcal{I}_h^1 \mathbf{u}_h \cdot \nabla \boldsymbol{\theta}_h - \pi_h^* (\mathcal{I}_h^1 \mathbf{u}_h \cdot \nabla \boldsymbol{\theta}_h) \right) \|_{0,\Omega} \\
&\leq \frac{1}{2\epsilon_2} \|h^{-\frac{1}{2}} \boldsymbol{\theta}^\pi\|_{0,\Omega}^2 + \frac{\epsilon_2 \gamma_1}{2} j_{\mathbf{u}_h}(\boldsymbol{\theta}_h, \boldsymbol{\theta}_h), \\
&\leq \frac{1}{2\epsilon_2} |\!|\!| (\boldsymbol{\theta}^\pi, 0) |\!|\!|^2 + \frac{\epsilon_2 \gamma_1}{2} |\!|\!| (\boldsymbol{\theta}_h, 0) |\!|\!|_{\mathbf{0}}^2.
\end{aligned}
$$

For the second term, we use approximation and that the divergence is included in the triple norm,

$$
\begin{aligned}
T_2 &= \frac{1}{2} (\boldsymbol{\theta}^\pi, (\nabla \cdot \boldsymbol{\theta}_h + \nabla \cdot \boldsymbol{\theta}^\pi) \boldsymbol{\theta}_h)) \\
&\leq \sum_{K\in\mathcal{T}_h} h_K^{-\frac{1}{2}} \|\boldsymbol{\theta}^\pi\|_{0,\infty,K} \left(\|h_K^{\frac{1}{2}} \nabla \cdot \boldsymbol{\theta}_h\|_{0,K} + \|h_K^{\frac{1}{2}} \nabla \cdot \boldsymbol{\theta}^\pi\|_{0,K} \right) \|\boldsymbol{\theta}_h\|_{0,K} \\
&\leq C_{14} \epsilon_3 h^{\frac{1}{2}} \|\nabla \mathbf{u}\|_{0,\infty,\Omega} \left(\|h^{\frac{1}{2}} \nabla \cdot \boldsymbol{\theta}_h\|_{0,\Omega}^2 + \|h^{\frac{1}{2}} \nabla \cdot \boldsymbol{\theta}^\pi\|_{0,\Omega}^2 \right) \\
&\quad + \frac{C_{14}}{\epsilon_3} h^{\frac{1}{2}} \|\nabla \mathbf{u}\|_{0,\infty,\Omega} \|\boldsymbol{\theta}_h\|_{0,\Omega}^2 \\
&\leq C_{15} \epsilon_3 \|\nabla \mathbf{u}\|_{0,\infty,\Omega} \left(|\!|\!| (\boldsymbol{\theta}_h, 0) |\!|\!|_{\mathbf{0}}^2 + h^{\frac{1}{2}} |\!|\!| (\boldsymbol{\theta}^\pi, 0) |\!|\!|^2 \right) \\
&\quad + \frac{C_{15}}{\epsilon_3} h^{\frac{1}{2}} \|\nabla \mathbf{u}\|_{0,\infty,\Omega} \|\boldsymbol{\theta}_h\|_{0,\Omega}^2.
\end{aligned}
$$

In this last inequality we used the fact that $0 < h \leq 1$.

For the third term, using (48), we have

$$\begin{aligned}
T_3 &= \frac{1}{2}\langle(\mathbf{u}_h - \mathbf{u}) \cdot \mathbf{n}\boldsymbol{\theta}^\pi, \boldsymbol{\theta}_h\rangle_{\partial\Omega} \\
&= -\frac{1}{2}\langle\boldsymbol{\theta}^\pi \cdot \mathbf{n}\boldsymbol{\theta}^\pi, \boldsymbol{\theta}_h\rangle_{\partial\Omega} - \frac{1}{2}\langle\boldsymbol{\theta}_h \cdot \mathbf{n}\boldsymbol{\theta}^\pi, \boldsymbol{\theta}_h\rangle_{\partial\Omega} \\
&\leq \frac{1}{2}\|\boldsymbol{\theta}^\pi\|_{0,\infty,\Omega}\|\boldsymbol{\theta}_h\|_{0,\partial\Omega}\left(\|\boldsymbol{\theta}^\pi \cdot \mathbf{n}\|_{0,\partial\Omega} + \|\boldsymbol{\theta}_h \cdot \mathbf{n}\|_{0,\partial\Omega}\right).
\end{aligned}$$

Therefore, using approximation and (25), we conclude that

$$\begin{aligned}
T_3 &\leq C_{16}h^{\frac{1}{2}}\|\nabla\mathbf{u}\|_{0,\infty,\Omega}\|\boldsymbol{\theta}_h\|_{0,\Omega}\left(|||(\boldsymbol{\theta}^\pi, 0)||| + |||(\boldsymbol{\theta}_h, 0)|||_{\mathbf{0}}\right) \\
&\leq C_{17}\frac{h}{\epsilon_4}\|\nabla\mathbf{u}\|^2_{0,\infty,\Omega}\|\boldsymbol{\theta}_h\|^2_{0,\Omega} + \epsilon_4 C_{17}\left(|||(\boldsymbol{\theta}^\pi, 0)|||^2 + |||(\boldsymbol{\theta}_h, 0)|||^2_{\mathbf{0}}\right).
\end{aligned}$$

Using again (48) and approximation, for the fourth term we obtain

$$\begin{aligned}
T_4 &= -((\boldsymbol{\theta}^\pi + \boldsymbol{\theta}_h) \cdot \nabla\mathbf{u}, \boldsymbol{\theta}_h) \\
&\leq \|\nabla\mathbf{u}\|_{0,\infty,\Omega}\left(\|\boldsymbol{\theta}^\pi\|_{0,\Omega} + \|\boldsymbol{\theta}_h\|_{0,\Omega}\right)\|\boldsymbol{\theta}_h\|_{0,\Omega} \\
&\leq \frac{1}{2}\|\nabla\mathbf{u}\|_{0,\infty,\Omega}\left(h^{\frac{1}{2}}\|h^{-\frac{1}{2}}\boldsymbol{\theta}^\pi\|^2_{0,\Omega} + 3\|\boldsymbol{\theta}_h\|^2_{0,\Omega}\right) \\
&\leq \frac{1}{2}\|\nabla\mathbf{u}\|_{0,\infty,\Omega}\left(h^{\frac{1}{2}}|||(\boldsymbol{\theta}^\pi, 0)|||^2 + 3\|\boldsymbol{\theta}_h\|^2_{0,\Omega}\right).
\end{aligned}$$

By testing (9) with $\mathbf{v}_h = \mathbf{0}$ and $q_h = \Pi_h^k(\mathbf{u} \cdot \boldsymbol{\theta}_h)$, it follows that

$$(\nabla \cdot \mathbf{u}_h, \Pi_h^k(\mathbf{u} \cdot \boldsymbol{\theta}_h)) - \langle\mathbf{u}_h \cdot \mathbf{n}, \Pi_h^k(\mathbf{u} \cdot \boldsymbol{\theta}_h)\rangle_{\partial\Omega} + j(p_h, \Pi_h^k(\mathbf{u} \cdot \boldsymbol{\theta}_h)) = 0.$$

Thus, plugging this expression into T_5, one gets

$$\begin{aligned}
T_5 &= \frac{1}{2}(\nabla \cdot \mathbf{u}_h, \mathbf{u} \cdot \boldsymbol{\theta}_h) \\
&= \frac{1}{2}(\nabla \cdot \mathbf{u}_h, (\mathbf{u} \cdot \boldsymbol{\theta}_h - \Pi_h^k(\mathbf{u} \cdot \boldsymbol{\theta}_h))) + \frac{1}{2}\langle\mathbf{u}_h \cdot \mathbf{n}, \Pi_h^k(\mathbf{u} \cdot \boldsymbol{\theta}_h)\rangle_{\partial\Omega} \\
&\quad - \frac{1}{2}j(p_h, \Pi_h^k(\mathbf{u} \cdot \boldsymbol{\theta}_h)),
\end{aligned}$$

which, from (48) and the fact that $\mathbf{u} = \mathbf{0}$ on $\partial\Omega$, leads to

$$
\begin{aligned}
T_5 = &-\underbrace{\frac{1}{2}(\nabla \cdot \boldsymbol{\theta}^\pi, \mathbf{u} \cdot \boldsymbol{\theta}_h - \Pi_h^k(\mathbf{u} \cdot \boldsymbol{\theta}_h))}_{T_{5,1}} \\
&-\underbrace{\frac{1}{2}(\nabla \cdot \boldsymbol{\theta}_h, \mathbf{u} \cdot \boldsymbol{\theta}_h - \Pi_h^k(\mathbf{u} \cdot \boldsymbol{\theta}_h))}_{T_{5,2}} \\
&-\underbrace{\frac{1}{2}\langle(\boldsymbol{\theta}^\pi + \boldsymbol{\theta}_h) \cdot \mathbf{n}, \Pi_h^k(\mathbf{u} \cdot \boldsymbol{\theta}_h) - \mathbf{u} \cdot \boldsymbol{\theta}_h\rangle_{\partial\Omega}}_{T_{5,3}} \\
&-\underbrace{\frac{1}{2}j(p_h, \Pi_h^k(\mathbf{u} \cdot \boldsymbol{\theta}_h))}_{T_{5,4}}.
\end{aligned}
$$

Each of these terms are treated separately. Using approximation and Corollary 4.1 we have

$$
\begin{aligned}
T_{5,1} &\leq \frac{1}{2}\|h^{\frac{1}{2}}\nabla \cdot \boldsymbol{\theta}^\pi\|_{0,\Omega}^2 + \frac{1}{2}\|h^{-\frac{1}{2}}(\mathbf{u} \cdot \boldsymbol{\theta}_h - \Pi_h^k(\mathbf{u} \cdot \boldsymbol{\theta}_h))\|_{0,\Omega}^2 \\
&\leq C_{18}\left(\|\!|(\boldsymbol{\theta}^\pi, 0)|\!\|^2 + h\|\mathbf{u}\|_{1,\infty,\Omega}^2\|\boldsymbol{\theta}_h\|_{0,\Omega}^2\right).
\end{aligned}
$$

Using again Corollary 4.1, it follows that

$$
\begin{aligned}
T_{5,2} &\leq \frac{1}{2}\|h^{\frac{1}{2}}\nabla \cdot \boldsymbol{\theta}_h\|_{0,\Omega}\|h^{-\frac{1}{2}}(\mathbf{u} \cdot \boldsymbol{\theta}_h - \Pi_h^k(\mathbf{u} \cdot \boldsymbol{\theta}_h))\|_{0,\Omega} \\
&\leq C_{19}\epsilon_5\|\!|(\boldsymbol{\theta}_h, 0)|\!\|_{\mathbf{0}}^2 + \frac{C_{19}h}{\epsilon_5}\|\mathbf{u}\|_{1,\infty,\Omega}^2\|\boldsymbol{\theta}_h\|_{0,\Omega}^2.
\end{aligned}
$$

For the next term, we have

$$
\begin{aligned}
T_{5,3} &\leq \frac{1}{2}\left(\|\boldsymbol{\theta}^{\pi}\cdot\mathbf{n}\|_{0,\partial\Omega}+\|\boldsymbol{\theta}_h\cdot\mathbf{n}\|_{0,\partial\Omega}\right)\|\Pi_h^k(\mathbf{u}\cdot\boldsymbol{\theta}_h)-\mathbf{u}\cdot\boldsymbol{\theta}_h\|_{0,\partial\Omega}\\
&\leq \frac{1}{2}\left(\|\!|(\boldsymbol{\theta}^{\pi},0)|\!\|_{\mathbf{0}}+\|\!|(\boldsymbol{\theta}_h,0)|\!\|_{\mathbf{0}}\right)\\
&\quad\left(\sum_{K\in\mathcal{T}_h}\|\Pi_h^k(\mathbf{u}\cdot\boldsymbol{\theta}_h)-\mathbf{u}\cdot\boldsymbol{\theta}_h\|^2_{0,\partial K\cap\partial\Omega}\right)^{\frac{1}{2}}\\
&\leq C_{20}\left(\|\!|(\boldsymbol{\theta}^{\pi},0)|\!\|+\|\!|(\boldsymbol{\theta}_h,0)|\!\|_{\mathbf{0}}\right)\\
&\quad\left[\sum_{K\in\mathcal{T}_h}\|\Pi_h^k(\mathbf{u}\cdot\boldsymbol{\theta}_h)-\Pi_h^k\mathbf{u}\cdot\boldsymbol{\theta}_h\|^2_{0,\partial K\cap\partial\Omega}\right.\\
&\quad\left.+\sum_{K\in\mathcal{T}_h}\|(\Pi_h^k\mathbf{u}-\mathbf{u})\cdot\boldsymbol{\theta}_h\|^2_{0,\partial K\cap\partial\Omega}\right]^{\frac{1}{2}}.
\end{aligned}
$$

Thus, using the trace inequality (25), we have

$$
\begin{aligned}
T_{5,3} &\leq C_{21}\left(\|\!|(\boldsymbol{\theta}^{\pi},0)|\!\|+\|\!|(\boldsymbol{\theta}_h,0)|\!\|_{\mathbf{0}}\right)\\
&\quad\left[\sum_{K\in\mathcal{T}_h}h_K^{-1}\|\Pi_h^k(\mathbf{u}\cdot\boldsymbol{\theta}_h)-\Pi_h^k\mathbf{u}\cdot\boldsymbol{\theta}_h\|^2_{0,K}\right.\\
&\quad\left.+\sum_{K\in\mathcal{T}_h}\|(\Pi_h^k\mathbf{u}-\mathbf{u})\|^2_{0,\infty,K}h_K^{-1}\|\boldsymbol{\theta}_h\|^2_{0,K}\right]^{\frac{1}{2}}\\
&\leq C_{22}h^{-\frac{1}{2}}\left(\|\!|(\boldsymbol{\theta}^{\pi},0)|\!\|+\|\!|(\boldsymbol{\theta}_h,0)|\!\|_{\mathbf{0}}\right)\\
&\quad\left(\|\Pi_h^k(\mathbf{u}\cdot\boldsymbol{\theta}_h)-\Pi_h^k\mathbf{u}\cdot\boldsymbol{\theta}_h\|_{0,\Omega}+\|(\Pi_h^k\mathbf{u}-\mathbf{u})\|_{0,\infty,\Omega}\|\boldsymbol{\theta}_h\|_{0,\Omega}\right).
\end{aligned}
$$

Now, using Corollary 4.1 and (22), we conclude that

$$
\begin{aligned}
T_{5,3} &\leq C_{23}\left(\|\!|(\boldsymbol{\theta}^{\pi},0)|\!\|+\|\!|(\boldsymbol{\theta}_h,0)|\!\|_{\mathbf{0}}\right)h^{\frac{1}{2}}\|\mathbf{u}\|_{1,\infty,\Omega}\|\boldsymbol{\theta}_h\|_{0,\Omega}\\
&\leq C_{24}\epsilon_6\left(\|\!|(\boldsymbol{\theta}^{\pi},0)|\!\|^2+\|\!|(\boldsymbol{\theta}_h,0)|\!\|^2_{\mathbf{0}}\right)+\frac{C_{24}h}{\epsilon_6}\|\mathbf{u}\|^2_{1,\infty,\Omega}\|\boldsymbol{\theta}_h\|^2_{0,\Omega}.
\end{aligned}
$$

For $T_{5,4}$ we first have,

$$
\begin{aligned}
T_{5,4} &= \sum_{K\in\mathcal{T}_h} \int_{\partial K} h_K^2 [\![\nabla p_h]\!] : [\![\nabla \Pi_h^k(\mathbf{u}\cdot\boldsymbol{\theta}_h)]\!] \,\mathrm{d}\boldsymbol{s} \\
&\le C_{25} j(p_h,p_h)^{\frac{1}{2}} \left[\sum_{K\in\mathcal{T}_h} \int_{\partial K} h_K^2 [\![\nabla(\Pi_h^k(\mathbf{u}\cdot\boldsymbol{\theta}_h) - \Pi_h^k\mathbf{u}\cdot\boldsymbol{\theta}_h)]\!]^2 \,\mathrm{d}\boldsymbol{s} \right.\\
&\quad \left. + \sum_{K\in\mathcal{T}_h} \int_{\partial K} h_K^2 [\![\nabla(\Pi_h^k\mathbf{u}\cdot\boldsymbol{\theta}_h)]\!]^2 \,\mathrm{d}\boldsymbol{s} \right]^{\frac{1}{2}} \\
&\le C_{26} j(p_h,p_h)^{\frac{1}{2}} \left[\sum_{K\in\mathcal{T}_h} h_K^2 \|\nabla(\Pi_h^k(\mathbf{u}\cdot\boldsymbol{\theta}_h) - \Pi_h^k\mathbf{u}\cdot\boldsymbol{\theta}_h)\|_{0,\partial K}^2 \right.\\
&\quad \left. + \sum_{K\in\mathcal{T}_h} \int_{\partial K} h_K^2 (|\boldsymbol{\theta}_h|^2 [\![\nabla(\Pi_h^k\mathbf{u})]\!]^2 + |\Pi_h^k\mathbf{u}|^2 [\![\nabla\boldsymbol{\theta}_h]\!]^2) \,\mathrm{d}\boldsymbol{s} \right]^{\frac{1}{2}}.
\end{aligned}
$$

Thus, by combining the trace inequality (25) with an inverse estimate (16), Corollary 4.1 and the stability estimate for the L^2-projection (20)-(21), we get

$$
\begin{aligned}
T_{5,4} &\le C_{27} j(p_h,p_h)^{\frac{1}{2}} \left[\sum_{K\in\mathcal{T}_h} h_K^{-1} \|\Pi_h^k(\mathbf{u}\cdot\boldsymbol{\theta}_h) - \Pi_h^k\mathbf{u}\cdot\boldsymbol{\theta}_h\|_{0,K}^2 \right.\\
&\quad \left. + \sum_{K\in\mathcal{T}_h} \|\nabla\Pi_h^k\mathbf{u}\|_{0,\infty,\Omega}^2 h_K \|\boldsymbol{\theta}_h\|_{0,K}^2 + \|\Pi_h^k\mathbf{u}\|_{0,\infty,\Omega}^2 j(\boldsymbol{\theta}_h,\boldsymbol{\theta}_h) \right]^{\frac{1}{2}} \\
&\le C_{28}\epsilon_7 (j(\Pi_h^k p, \Pi_h^k p) + j(y_h,y_h)) + \frac{C_{28}}{\epsilon_7} h \|\mathbf{u}\|_{1,\infty,\Omega}^2 \|\boldsymbol{\theta}_h\|_{0,\Omega}^2 \\
&\quad + \frac{C_\pi^2}{2\epsilon_7} \|\mathbf{u}\|_{0,\infty,\Omega}^2 j(\boldsymbol{\theta}_h,\boldsymbol{\theta}_h) \\
&\le C_{28}\epsilon_7 \left(j(\Pi_h^k p, \Pi_h^k p) + |\!|\!|(\mathbf{0},y_h)|\!|\!|_{\mathbf{0}} \right) + \frac{C_{28}}{\epsilon_7} h \|\mathbf{u}\|_{1,\infty,\Omega}^2 \|\boldsymbol{\theta}_h\|_{0,\Omega}^2 \\
&\quad + \frac{C_\pi^2}{2\epsilon_7} \|\mathbf{u}\|_{0,\infty,\Omega}^2 j(\boldsymbol{\theta}_h,\boldsymbol{\theta}_h).
\end{aligned}
$$

Finally, for the last term in (49), using (48), we have

$$
\begin{aligned}
T_6 &= \sum_{K\in\mathcal{T}_h} \int_{\partial K} h_K^2 |\mathcal{I}_h^1 \mathbf{u}_h \cdot \mathbf{n}|^2 [\![\nabla \Pi_h^k \mathbf{u}]\!] : [\![\nabla \boldsymbol{\theta}_h]\!] \,\mathrm{d}\boldsymbol{s} \\
&\leq \frac{1}{2\epsilon_8} \sum_{K\in\mathcal{T}_h} \int_{\partial K} h_K^2 |\mathcal{I}_h^1 \mathbf{u}_h|^2 [\![\nabla \Pi_h^k \mathbf{u}]\!]^2 \,\mathrm{d}\boldsymbol{s} \\
&\quad + \frac{\epsilon_8}{2} \sum_{K\in\mathcal{T}_h} \int_{\partial K} h_K^2 |\mathcal{I}_h^1 \mathbf{u}_h \cdot \mathbf{n}|^2 [\![\nabla \boldsymbol{\theta}_h]\!]^2 \,\mathrm{d}\boldsymbol{s} \\
&\leq \frac{C_{29}}{2\epsilon_8} \Big[\underbrace{\sum_{K\in\mathcal{T}_h} \int_{\partial K} h_K^2 |\mathcal{I}_h^1 \mathbf{u}_h - \mathbf{u}_h|^2 [\![\nabla \Pi_h^k \mathbf{u}]\!]^2 \,\mathrm{d}\boldsymbol{s}}_{T_{6,1}} \\
&\qquad + \underbrace{\sum_{K\in\mathcal{T}_h} \int_{\partial K} h_K^2 \left(|\boldsymbol{\theta}_h|^2 + |\Pi_h^k \mathbf{u}|^2 \right) [\![\nabla \Pi_h^k \mathbf{u}]\!]^2 \,\mathrm{d}\boldsymbol{s} \Big]}_{T_{6,2}} \\
&\quad + \frac{\epsilon_8}{2} \sum_{K\in\mathcal{T}_h} \int_{\partial K} h_K^2 |\mathcal{I}_h^1 \mathbf{u}_h \cdot \mathbf{n}|^2 [\![\nabla \boldsymbol{\theta}_h]\!]^2 \,\mathrm{d}\boldsymbol{s}.
\end{aligned}
\tag{50}
$$

In order to estimate $T_{6,1}$, we use approximation and an inverse inequality to obtain

$$
\begin{aligned}
\|\mathcal{I}_h^1 \mathbf{u}_h - \mathbf{u}_h\|_{0,\partial K}^2 &\leq C_{30} |\mathbf{u}|_{2,\partial K}^2 \\
&= C_{30} |\mathcal{I}_h^1 \Pi_h^k \mathbf{u} - \mathbf{u}_h|_{2,\partial K}^2 \\
&\leq C_{31} \|\mathcal{I}_h^1 \Pi_h^k \mathbf{u} - \mathbf{u}_h\|_{0,\partial K}^2.
\end{aligned}
$$

This estimate, combined with (21), the trace inequality (25), approximation and an inverse inequality (16), leads to

$$\begin{aligned}
&\int_{\partial K} h_K^2 |\mathcal{I}_h^1 \mathbf{u}_h - \mathbf{u}_h|^2 [\![\nabla \Pi_h^k \mathbf{u}]\!]^2 \,\mathrm{d}\boldsymbol{s} \\
&\leq \|[\![\nabla \Pi_h^k \mathbf{u}]\!]\|_{0,\infty,K}^2 \int_{\partial K} h_K^2 |\mathcal{I}_h^1 \mathbf{u}_h - \mathbf{u}_h|^2 \,\mathrm{d}\boldsymbol{s} \\
&\leq C_{32} \|[\![\nabla \Pi_h^k \mathbf{u}]\!]\|_{0,\infty,K}^2 \int_{\partial K} h_K^2 |\mathcal{I}_h^1 \Pi_h^k \mathbf{u} - \mathbf{u}_h|^2 \,\mathrm{d}\boldsymbol{s} \\
&\leq C_{33} \|[\![\nabla \Pi_h^k \mathbf{u}]\!]\|_{0,\infty,K}^2 \int_{\partial K} h_K^2 \left(|\mathcal{I}_h^1 \Pi_h^k \mathbf{u} - \Pi_h^k \mathbf{u}|^2 + |\boldsymbol{\theta}_h|^2 \right) \,\mathrm{d}\boldsymbol{s} \\
&\leq C_{34} h_K \|[\![\nabla \Pi_h^k \mathbf{u}]\!]\|_{0,\infty,K}^2 \left(\|\mathcal{I}_h^1 \Pi_h^k \mathbf{u} - \Pi_h^k \mathbf{u}\|_{0,K}^2 + \|\boldsymbol{\theta}_h\|_{0,K}^2 \right) \\
&\leq C_{35} h_K \|[\![\nabla \Pi_h^k \mathbf{u}]\!]\|_{0,\infty,K}^2 \left(h_K^d \|\mathcal{I}_h^1 \Pi_h^k \mathbf{u} - \Pi_h^k \mathbf{u}\|_{0,\infty,K}^2 + \|\boldsymbol{\theta}_h\|_{0,K}^2 \right) \\
&\leq C_{36} h_K \|[\![\nabla \Pi_h^k \mathbf{u}]\!]\|_{0,\infty,K}^2 \left(h^{d+2} \|\Pi_h^k \mathbf{u}\|_{1,\infty,\Omega}^2 + \|\boldsymbol{\theta}_h\|_{0,K}^2 \right) \\
&\leq C_{37} h_K^{2-d} \|[\![\nabla \Pi_h^k \mathbf{u}]\!]\|_{0,\partial K}^2 h^{d+2} \|\mathbf{u}\|_{1,\infty,\Omega}^2 + h_K C_{37} \|\nabla \mathbf{u}\|_{1,\infty,\Omega}^2 \|\boldsymbol{\theta}_h\|_{0,K}^2.
\end{aligned}$$

Therefore,

$$T_{6,1} \leq C_{38} \|\mathbf{u}\|_{1,\infty,\Omega}^2 \left(h^2 j(\Pi_h^k \mathbf{u}, \Pi_h^k \mathbf{u}) + h \|\boldsymbol{\theta}_h\|_{0,\Omega}^2 \right).$$

Using the trace inequality (25) and the stability of the L^2-projection (21), we obtain

$$T_{6,2} \leq h \|\mathbf{u}\|_{1,\infty,\Omega}^2 \|\boldsymbol{\theta}_h\|_{0,\Omega}^2 + \|\mathbf{u}\|_{0,\infty,\Omega}^2 j(\Pi_h^k \mathbf{u}, \Pi_h^k \mathbf{u}).$$

Finally, by plugging these last two estimates into (50), we obtain

$$\begin{aligned}
T_6 \leq{}& \frac{C_{39}}{\epsilon_8} \left[h \|\mathbf{u}\|_{1,\infty,\Omega}^2 \|\boldsymbol{\theta}_h\|_{0,\Omega}^2 + (\|\mathbf{u}\|_{0,\infty,\Omega}^2 + h^2 \|\mathbf{u}\|_{1,\infty,\Omega}^2) j(\Pi_h^k \mathbf{u}, \Pi_h^k \mathbf{u}) \right] \\
&+ \frac{\epsilon_8}{2} |\!|\!|(\boldsymbol{\theta}_h, 0)|\!|\!|_{\mathbf{u}_h}^2.
\end{aligned}$$

Based on (49) and the previous estimates, we chose ϵ_i, $i = 1, \ldots, 8$, such that

$$\begin{aligned}
&\frac{\epsilon_1 C_1}{2} = \frac{\epsilon_2 \gamma_1}{2} = \epsilon_3 C_{15} \|\nabla \mathbf{u}\|_{0,\infty,\Omega} = \epsilon_4 C_{17} = e_5 C_{19} = \epsilon C_{24} = \frac{C_{\mathrm{A}}}{16}, \\
&\epsilon_7 C_{28} = \frac{\epsilon_8}{2} = \frac{C_{\mathrm{A}}}{16},
\end{aligned}$$

and γ such as

$$\gamma > \frac{C_\pi^2}{2\epsilon_7} \|\mathbf{u}\|_{0,\infty,\Omega}^2,$$

for instance

$$\gamma \stackrel{\text{def}}{=} \frac{8 C_\pi^2}{C_{\mathrm{A}}} \|\mathbf{u}\|_{0,\infty,\Omega}^2 + 1.$$

Then, from (49) using the previous estimates, we get

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{\theta}_h\|_{0,2}^2 &+ \frac{C_{\mathrm{A}}}{2}|\!|\!|(\boldsymbol{\theta}_h, y_h)|\!|\!|_{\mathbf{u}_h}^2 \\ &\leq C_{40}\left(h\|\mathbf{u}\|_{1,\infty,\Omega}^2 + \|\mathbf{u}\|_{1,\infty,\Omega}\right)\|\boldsymbol{\theta}_h\|_{0,\Omega}^2 \\ &\quad + C_{40}|\!|\!|(\boldsymbol{\theta}^\pi, 0)|\!|\!|^2\left(1 + h^{\frac{1}{2}}\|\nabla\mathbf{u}\|_{0,\infty,\Omega}\right) \\ &\quad + C_{40}(1 + \|\mathbf{u}\|_{0,\infty,\Omega}^2 + h^2\|\mathbf{u}\|_{1,\infty,\Omega}^2)\mathbf{J}\left[\mathbf{0}; (\Pi_h^k\mathbf{u}, \Pi_h^k p), (\Pi_h^k\mathbf{u}, \Pi_h^k p)\right], \end{aligned} \tag{51}$$

a.e. in $(0, T)$. Therefore, using Gronwall's lemma we obtain

$$\begin{aligned}\|\boldsymbol{\theta}_h\|_{L^\infty(0,T;L^2(\Omega))}^2 &+ \int_0^T |\!|\!|(\boldsymbol{\theta}_h, y_h)|\!|\!|_{\mathbf{u}_h}^2 \,\mathrm{d}t \leq \|\boldsymbol{\theta}_h(0)\|_{0,\Omega}^2 \\ &+ C_{\exp}\left[c_1\int_0^T |\!|\!|(\boldsymbol{\theta}^\pi, 0)|\!|\!|^2 \,\mathrm{d}t + c_2\int_0^T \mathbf{J}\left[\mathbf{0}; (\Pi_h^k\mathbf{u}, \Pi_h^k p), (\Pi_h^k\mathbf{u}, \Pi_h^k p)\right]\mathrm{d}t\right],\end{aligned}$$

with

$$\begin{aligned} C_{\exp} &\stackrel{\mathrm{def}}{=} e^{C_{40}T\left(h\|\mathbf{u}\|_{L^\infty(0,T;W^{1,\infty}(\Omega))}^2 + \|\mathbf{u}\|_{L^\infty(0,T;W^{1,\infty}(\Omega))}\right)}, \\ c_1 &\stackrel{\mathrm{def}}{=} C_{40}\left(1 + h^{\frac{1}{2}}\|\mathbf{u}\|_{L^\infty(0,T;W^{1,\infty}(\Omega))}\right), \\ c_2 &\stackrel{\mathrm{def}}{=} C_{40}\left(1 + \|\mathbf{u}\|_{L^\infty(0,T;L^\infty(\Omega))}^2 + h^2\|\mathbf{u}\|_{L^\infty(0,T;W^{1,\infty}(\Omega))}^2\right), \end{aligned}$$

which gives (45). Finally, estimate (46) is obtained using (51) and integration over $(0, T)$.

6.3 Pressure estimates

Based on the previous convergence analysis for the velocity, in this paragraph, we provide error estimates for the pressure. The optimal approach to follow here is not clear cut. Depending on how much regularity one can expect for the pressure, different analysis should be applied. Here we choose first to present two estimates for the pressure in the L^2-norm and, for regular pressures, in the H^1-norm. The upper bounds consist of one part using the previous convergence of the velocities, and a second part consisting of different norms of the approximation error in the time derivative of the velocities. We will then show how to get an estimate that is optimal for low-order elements if the pressure is in $H^1(0, T; H^1(\Omega))$. Other possible strategies will be briefly discussed in the conclusion.

Lemma 6.2 *Let* $(\mathbf{u}, p)$ *the solution of* (1), $(\mathbf{u}_h, p_h) \in W_h^k$ *the solution of* (9), *with* $\mathbf{u}_{0,h} = P_h^k\mathbf{u}_0$, *and assume that* $(\mathbf{u}, p)$ *has the regularity* (5) *and that* $\nu < h$. *Then, the following* *error estimate holds*

$$\int_0^T \|p - p_h\|_{0,\Omega}^2 \,\mathrm{d}t \le C\Big[h^{2(r_{\mathbf{u}}-1)}\|\mathbf{u}\|^2_{L^2(0,T;H^{r_{\mathbf{u}}}(\Omega))} \\ + h^{2(r_p-1)}\|p\|^2_{L^2(0,T;H^{r_p}(\Omega))} \\ + \|\partial_t(\mathbf{u}-\mathbf{u}_h)\|^2_{L^2(0,T;V'(\Omega))}\Big].$$

with V' standing for the dual of $H^1(\Omega)$ and $C > 0$ a constant independent of ν and h.

Proof. Following [20, Corollary 2.4], there exists $\mathbf{v}_p \in [H_0^1(\Omega)]^d$ such that

$$\nabla \cdot \mathbf{v}_p = p - p_h, \quad \|\mathbf{v}_p\|_{1,\Omega} \le C\|p - p_h\|_{0,\Omega}. \tag{52}$$

In particular, we have

$$\|p-p_h\|^2_{0,\Omega} = (p-p_h, \nabla\cdot\mathbf{v}_p) \\ = (p-p_h, \nabla\cdot(\mathbf{v}_p - \Pi_h^k\mathbf{v}_p)) + (p-p_h, \nabla\cdot(\Pi_h^k\mathbf{v}_p)).$$

After integration by parts and using (13), we obtain

$$\|p-p_h\|^2_{0,\Omega} = -(\nabla(p-p_h), \mathbf{v}_p - \Pi_h^k\mathbf{v}_p) - \langle p-p_h, (\Pi_h^k\mathbf{v}_p)\cdot\mathbf{n}\rangle_{\partial\Omega} \\ + (p-p_h, \nabla\cdot(\Pi_h^k\mathbf{v}_p)) \\ = -(\nabla(p-p_h), \mathbf{v}_p - \Pi_h^k\mathbf{v}_p) - b_h(p-p_h, \Pi_h^k\mathbf{v}_p).$$

Therefore, using the approximate Galerkin orthogonality (Lemma 3.1) with $(\mathbf{v}_h, q_h) = (\Pi_h^k\mathbf{v}_p, 0)$, we get

$$\|p-p_h\|^2_{0,\Omega} = -\underbrace{(\nabla(p-p_h), \mathbf{v}_p - \Pi_h^k\mathbf{v}_p)}_{T_1} \\ + \underbrace{a_h(\mathbf{u}-\mathbf{u}_h, \Pi_h^k\mathbf{v}_p) - \gamma j(\mathbf{u}_h, \Pi_h^k\mathbf{v}_p)}_{T_2} \\ + \underbrace{c_h(\mathbf{u};\mathbf{u},\Pi_h^k\mathbf{v}_p) - c_h(\mathbf{u}_h;\mathbf{u}_h,\Pi_h^k\mathbf{v}_p)}_{T_3} \\ - \underbrace{j_{\mathbf{u}_h}(\mathbf{u}_h, \Pi_h^k\mathbf{v}_p)}_{T_4} + \underbrace{(\partial_t(\mathbf{u}-\mathbf{u}_h), \Pi_h^k\mathbf{v}_p)}_{T_5}. \tag{53}$$

Terms T_1 and T_2 are treated using the arguments given in [10, Proof of Theorem 4.4]. Thus, for the first term, using the orthogonality of the L^2-projection, Cauchy-Schwarz, (28), (19) and since $p \in H^s(\Omega)$, we get

$$\begin{aligned} T_1 &= \big(\nabla p - \Pi_h^k(\nabla p) - (\nabla p_h - \pi_h^*(\nabla p_h)), \mathbf{v}_p - \Pi_h^k \mathbf{v}_p\big) \\ &\leq \Big(\|\nabla p - \Pi_h^k(\nabla p)\|_{0,\Omega} + h^{-\frac{1}{2}} \|h^{\frac{1}{2}}(\nabla p_h - \pi_h^*(\nabla p_h))\|_{0,\Omega}\Big) \\ &\quad \times \|\mathbf{v}_p - \Pi_h^k \mathbf{v}_p\|_{0,\Omega} \\ &\leq C \left(\|\nabla p - \Pi_h^k(\nabla p)\|_{0,\Omega} + h^{-\frac{1}{2}} j(p_h,p_h)^{\frac{1}{2}}\right) h \|\mathbf{v}_p\|_{1,\Omega} \\ &\leq C \left(C h^{r_p} \|p\|_{r_p,\Omega} + h^{\frac{1}{2}} |\!|\!|(\mathbf{u}-\mathbf{u}_h, p - p_h)|\!|\!|_{\mathbf{u}_h}\right) \|\mathbf{v}_p\|_{1,\Omega}. \end{aligned}$$

For the second term we have

$$\begin{aligned} T_2 \leq{} & C |\!|\!|(\mathbf{u}-\mathbf{u}_h, 0)|\!|\!|_{\mathbf{0}} |\!|\!|(\Pi_h^k \mathbf{v}_p, 0)|\!|\!|_{\mathbf{0}} \\ & \underbrace{- \langle 2\nu \boldsymbol{\varepsilon}(\mathbf{u}-\mathbf{u}_h)\mathbf{n}, \Pi_h^k \mathbf{v}_p \rangle_{\partial\Omega} - \langle \mathbf{u}-\mathbf{u}_h, 2\nu \boldsymbol{\varepsilon}(\Pi_h^k \mathbf{v}_p)\mathbf{n} \rangle_{\partial\Omega}}_{T_{2,1}}. \end{aligned}$$

The boundary terms are controlled in the following fashion

$$\begin{aligned} T_{2,1} &\leq 2 \|(\nu h)^{\frac{1}{2}} \boldsymbol{\varepsilon}(\mathbf{u}-\mathbf{u}_h)\|_{0,\partial\Omega} \|\nu^{\frac{1}{2}} h^{-\frac{1}{2}} \Pi_h^k \mathbf{v}_p\|_{0,\partial\Omega} \\ &\quad + 2 \|(\nu h)^{\frac{1}{2}} \boldsymbol{\varepsilon}(\Pi_h^k \mathbf{v}_p)\|_{0,\partial\Omega} \|\nu^{\frac{1}{2}} h^{-\frac{1}{2}} (\mathbf{u}-\mathbf{u}_h)\|_{0,\partial\Omega} \\ &\leq 2 \|(\nu h)^{\frac{1}{2}} \boldsymbol{\varepsilon}(\mathbf{u}-\mathbf{u}_h)\|_{0,\partial\Omega} |\!|\!|(\Pi_h^k \mathbf{v}_p, 0)|\!|\!|_{\mathbf{0}} \\ &\quad + 2 \|(\nu h)^{\frac{1}{2}} \boldsymbol{\varepsilon}(\Pi_h^k \mathbf{v}_p)\|_{0,\partial\Omega} |\!|\!|(\mathbf{u}-\mathbf{u}_h, 0)|\!|\!|_{\mathbf{0}}. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} \|(\nu h)^{\frac{1}{2}} \boldsymbol{\varepsilon}(\mathbf{u}-\mathbf{u}_h)\|_{0,\partial\Omega} \leq{} & \|(\nu h)^{\frac{1}{2}} \boldsymbol{\varepsilon}(\mathbf{u}-\Pi_h^k \mathbf{u})\|_{0,\partial\Omega} \\ & + \|(\nu h)^{\frac{1}{2}} \boldsymbol{\varepsilon}(\Pi_h^k \mathbf{u}-\mathbf{u}_h)\|_{0,\partial\Omega}, \end{aligned}$$

where the first term satisfies, using the trace inequality (24), (19) and that $\nu < h$,

$$\begin{aligned} \|(\nu h)^{\frac{1}{2}} \boldsymbol{\varepsilon}(\mathbf{u}-\Pi_h^k \mathbf{u})\|_{0,\partial\Omega} &\leq C \nu^{\frac{1}{2}} h^{r_{\mathbf{u}}-1} \|\mathbf{u}\|_{r_{\mathbf{u}},\Omega} \\ &\leq C h^{r_{\mathbf{u}}-\frac{1}{2}} \|\mathbf{u}\|_{r_{\mathbf{u}},\Omega}, \end{aligned}$$

and the second, using (25),

$$\begin{aligned} \|(\nu h)^{\frac{1}{2}} \boldsymbol{\varepsilon}(\Pi_h^k \mathbf{u}-\mathbf{u}_h)\|_{0,\partial\Omega} &\leq C \|\nu^{\frac{1}{2}} \boldsymbol{\varepsilon}(\Pi_h^k \mathbf{u}-\mathbf{u}_h)\|_{0,\Omega} \\ &\leq C |\!|\!|(\Pi_h^k \mathbf{u}-\mathbf{u}_h, 0)|\!|\!|_{\mathbf{0}}. \end{aligned}$$

In the same fashion we conclude that

$$\|(\nu h)^{\frac{1}{2}} \boldsymbol{\varepsilon}(\Pi_h^k \mathbf{v}_p)\|_{0,\partial\Omega} \leq C |\!|\!|(\Pi_h^k \mathbf{v}_p, 0)|\!|\!|_{\mathbf{0}}.$$

Thus, collecting terms, using the fact that

$$|||(\Pi_h^k \mathbf{v}_p, 0)|||_{\mathbf{0}} \leq C\|\Pi_h^k \mathbf{v}_p\|_{1,\Omega},$$

and the H^1-stability of the L^2-projection (18), we have

$$T_2 \leq C\left(|||(\mathbf{u}-\mathbf{u}_h, 0)|||_{\mathbf{0}} + h^{r_{\mathbf{u}}-\frac{1}{2}}\|\mathbf{u}\|_{r_{\mathbf{u}},\Omega}\right)\|\mathbf{v}_p\|_{1,\Omega}.$$

For the third term, we have

$$\begin{aligned} T_3 &= (\mathbf{u}\cdot\nabla(\mathbf{u}-\mathbf{u}_h), \Pi_h^k \mathbf{v}_p) + ((\mathbf{u}-\mathbf{u}_h)\cdot\nabla\mathbf{u}_h, \Pi_h^k \mathbf{v}_p) \\ &\quad + \frac{1}{2}\langle \mathbf{u}_h\cdot\mathbf{n}\mathbf{u}_h, \Pi_h^k \mathbf{v}_p\rangle_{\partial\Omega} - \frac{1}{2}(\nabla\cdot\mathbf{u}_h, \mathbf{u}_h\cdot\Pi_h^k \mathbf{v}_p). \end{aligned}$$

Thus, integrating by parts and since $\nabla\cdot\mathbf{u} = 0$ and $\mathbf{u}_{|\partial\Omega} = \mathbf{0}$, we get

$$\begin{aligned} T_3 &= \underbrace{-((\mathbf{u}-\mathbf{u}_h)\cdot\nabla\Pi_h^k \mathbf{v}_p, \mathbf{u}) - (\mathbf{u}_h\cdot\nabla\Pi_h^k \mathbf{v}_p, \mathbf{u}-\mathbf{u}_h)}_{T_{3,1}} \\ &\quad - \frac{1}{2}\underbrace{\langle \mathbf{u}_h\cdot\mathbf{n}\mathbf{u}_h, \Pi_h^k \mathbf{v}_p\rangle_{\partial\Omega}}_{T_{3,2}} + \frac{1}{2}\underbrace{(\nabla\cdot\mathbf{u}_h, \mathbf{u}_h\cdot\Pi_h^k \mathbf{v}_p)}_{T_{3,3}}. \end{aligned} \tag{54}$$

Now, we treat each term separately. Using Cauchy-Schwarz on the first term followed by (18), one gets

$$\begin{aligned} T_{3,1} &\leq C\left(\|\mathbf{u}\|_{0,\infty,\Omega} + \|\mathbf{u}_h\|_{0,\infty,\Omega}\right)\|\mathbf{u}-\mathbf{u}_h\|_{0,\Omega}\|\nabla\Pi_h^k \mathbf{v}_p\|_{0,\Omega} \\ &\leq C\left(\|\mathbf{u}\|_{0,\infty,\Omega} + \|\mathbf{u}_h\|_{0,\infty,\Omega}\right)\|\mathbf{u}-\mathbf{u}_h\|_{0,\Omega}\|\mathbf{v}_p\|_{1,\Omega}. \end{aligned}$$

For the second, using Cauchy-Schwarz and the argument followed in (33), we obtain

$$\begin{aligned} T_{3,2} &\leq \|\mathbf{u}_h\|_{0,\infty,\Omega}\|\mathbf{u}_h\cdot\mathbf{n}\|_{0,\partial\Omega}\|\Pi_h^k \mathbf{v}_p - \mathbf{v}_p\|_{0,\partial\Omega} \\ &\leq Ch^{\frac{1}{2}}\|\mathbf{u}_h\|_{0,\infty,\Omega}|||(\mathbf{u}-\mathbf{u}_h, p-p_h)|||_{\mathbf{u}_h}\|\mathbf{v}_p\|_{1,\Omega}. \end{aligned}$$

Using Cauchy-Schwarz and the stability of the L^2-projection, in the third term, we have

$$\begin{aligned} T_{3,3} &\leq h^{-\frac{1}{2}}\|h^{\frac{1}{2}}\nabla\cdot\mathbf{u}_h\|_{0,\Omega}\|\mathbf{u}_h\|_{0,\infty,\Omega}\|\Pi_h^k \mathbf{v}_p\|_{0,\Omega} \\ &\leq Ch^{-\frac{1}{2}}\|\mathbf{u}_h\|_{0,\infty,\Omega}|||(\mathbf{u}-\mathbf{u}_h, p-p_h)|||_{\mathbf{u}_h}\|\mathbf{v}_p\|_{1,\Omega}. \end{aligned} \tag{55}$$

For the jump term in (53), we use Cauchy-Schwarz, (16), the H^1–stability of the L^2–projection (18) and the regularity $\mathbf{u} \in H^r(\Omega)$, which yields

$$
\begin{aligned}
T_4 &\leq j_{\mathbf{u}_h}(\mathbf{u}_h, \mathbf{u}_h)^{\frac{1}{2}} j_{\mathbf{u}_h}(\Pi_h^k \mathbf{v}_p, \Pi_h^k \mathbf{v}_p)^{\frac{1}{2}} \\
&\leq j_{\mathbf{u}_h}(\mathbf{u}_h, \mathbf{u}_h)^{\frac{1}{2}} \left(\sum_{K \in \mathcal{T}_h} \int_{\partial K} h_K^2 |\mathbf{u}_h \cdot \mathbf{n}|^2 [\![\nabla \Pi_h^k \mathbf{v}_p]\!]^2 \right)^{\frac{1}{2}} \\
&\leq C j_{\mathbf{u}_h}(\mathbf{u}_h, \mathbf{u}_h)^{\frac{1}{2}} \|\mathbf{u}_h\|_{0,\infty,\Omega} \left(\sum_{K \in \mathcal{T}_h} h_K \|\nabla \Pi_h^k \mathbf{v}_p\|_{0,K}^2 \right)^{\frac{1}{2}} \\
&\leq C h^{\frac{1}{2}} j_{\mathbf{u}_h}(\mathbf{u}_h, \mathbf{u}_h)^{\frac{1}{2}} \|\mathbf{u}_h\|_{0,\infty,\Omega} \|\mathbf{v}_p\|_{1,\Omega}, \\
&\leq C h^{\frac{1}{2}} \|\mathbf{u}_h\|_{0,\infty,\Omega} |\!|\!|(\mathbf{u} - \mathbf{u}_h, p - p_h)|\!|\!|_{\mathbf{u}_h} \|\mathbf{v}_p\|_{1,\Omega}.
\end{aligned}
$$

Finally by duality and the H^1-stability of the L^2-projection we have

$$
T_5 \leq \|\partial_t(\mathbf{u} - \mathbf{u}_h)\|_{V'} \|\mathbf{v}_p\|_{1,\Omega}.
$$

Therefore, from (52) a Young's inequality and by collecting the previous estimations, we have

$$
\begin{aligned}
\|p - p_h\|_{0,\Omega}^2 \leq C \Big\{ & h^{2r_p} \|p\|_{r_p,\Omega}^2 + h^{2r_{\mathbf{u}}-1} \|\mathbf{u}\|_{r_{\mathbf{u}},\Omega}^2 \\
& + \left[1 + h + (h + h^{-1}) \|\mathbf{u}_h\|_{0,\infty,\Omega}^2\right] |\!|\!|(\mathbf{u} - \mathbf{u}_h, p - p_h)|\!|\!|_{\mathbf{u}_h}^2 \\
& + \left(\|\mathbf{u}\|_{0,\infty,\Omega}^2 + \|\mathbf{u}_h\|_{0,\infty,\Omega}^2 \right) \|\mathbf{u} - \mathbf{u}_h\|_{0,\Omega}^2 \Big\}.
\end{aligned}
$$

We conclude the proof after integration over $(0, T)$ and application of the results of Corollaries 6.1 and 6.2. □

Remark 6.1 *From the optimal convergence estimate provided by Corollary 6.1, one would expect a similar rate for the pressure. However, the fact that, at the discrete level, the convective velocity is not divergence free, leads to a loss of half an order in the pressure estimate, see equation* (55).

Unfortunately, it seems difficult to obtain an estimate of the time derivative in the dual norm of H^1 (appearing in the previous Lemma). As pointed out in [25], one may obtain a crude bound by using, instead, the L^2-norm of the error in the time derivative. We therefore propose to estimate the error in the gradients of the pressure, valid only if the pressure is sufficiently regular, but leading to an estimate which is close to optimal.

Lemma 6.3 *Under the hypothesis of the previous corollary, the following error estimate holds*

$$\int_0^T \|\nabla(p - p_h)\|_{0,\Omega}^2 \,\mathrm{d}t \leq \Big[\Big(h^{2r_{\mathbf{u}}-3} + h^{2(r_{\mathbf{u}}-1)}\Big) \|\mathbf{u}\|^2_{L^2(0,T;H^{r_{\mathbf{u}}}(\Omega))} + \Big(h^{2r_p-3} + h^{2(r_p-1)}\Big) \|p\|^2_{L^2(0,T;H^{r_p}(\Omega))} + \|\partial_t(\mathbf{u} - \mathbf{u}_h)\|^2_{L^2(0,T;L^2(\Omega))} \Big].$$

Proof. To estimate the error in the pressure gradient we start by noticing that

$$\int_0^T \|\nabla(p - p_h)\|_{0,\Omega}^2 \,\mathrm{d}t \leq C \int_0^T \|\nabla p - \Pi_h^k(\nabla p)\|^2_{0,\Omega} \,\mathrm{d}t + C \int_0^T \|\Pi_h^k(\nabla p) - \Pi_h^k(\nabla p_h)\|^2_{0,\Omega} \,\mathrm{d}t + C \int_0^T \|\Pi_h^k(\nabla p_h) - \nabla p_h\|^2_{0,\Omega} \,\mathrm{d}t.$$

Using approximation, (28), the fact that $p \in H^s(\Omega)$ and Corollary 6.1, we obtain

$$\begin{aligned}\int_0^T \|\nabla p - \Pi_h^k(\nabla p)\|^2_{0,\Omega} + \int_0^T \|\Pi_h^k(\nabla p_h) - \nabla p_h\|^2_{0,\Omega} &\leq Ch^{2(r_p-1)} \|p\|^2_{L^\infty(0,T;H^{r_p}(\Omega))} + Ch^{-1} \int_0^T j(p_h, p_h) \,\mathrm{d}t \\ &= Ch^{2(r_p-1)} \|p\|^2_{L^\infty(0,T;H^{r_p}(\Omega))} + Ch^{-1} \int_0^T j(p - p_h, p - p_h) \,\mathrm{d}t \\ &\leq C\Big[h^{2(r_{\mathbf{u}}-1)} \|\mathbf{u}\|^2_{L^2(0,T;H^{r_{\mathbf{u}}}(\Omega))} + h^{2(r_p-1)} \|p\|^2_{L^2(0,T;H^{r_p}(\Omega))}\Big]. \end{aligned} \tag{56}$$

Hence it is sufficient to study the second term. Note that, by the orthogonality of the L^2-projection and a partial integration, we have

$$\begin{aligned} \|\Pi_h^k(\nabla p) - \Pi_h^k(\nabla p_h)\|^2_{0,\Omega} &= (\nabla p - \nabla p_h, \Pi_h^k(\nabla p) - \Pi_h^k(\nabla p_h)) \\ &= -(p - p_h, \nabla \cdot (\Pi_h^k(\nabla p) - \Pi_h^k(\nabla p_h))) \\ &\quad + \langle p - p_h, (\Pi_h^k(\nabla p) - \Pi_h^k(\nabla p_h)) \cdot \mathbf{n} \rangle_{\partial\Omega} \\ &= b_h(p - p_h, \mathbf{w}^k_{h,p}), \end{aligned}$$

with the notation $\mathbf{v}^k_{h,p} \stackrel{\mathrm{def}}{=} \Pi_h^k(\nabla p - \nabla p_h)$.

Therefore, using the approximate Galerkin orthogonality (Lemma 3.1) with $(\mathbf{v}_h, q_h) = (\mathbf{v}^k_{h,p}, 0)$, we get

$$\begin{aligned}\|\Pi_h^k \nabla p - \Pi_h^k \nabla p_h\|^2_{0,\Omega} = a_h(\mathbf{u}-\mathbf{u}_h, \mathbf{v}^k_{h,p}) - \gamma j(\mathbf{u}_h, \mathbf{v}^k_{h,p}) \\ + c_h(\mathbf{u};\mathbf{u}, \mathbf{v}^k_{h,p}) - c_h(\mathbf{u}_h;\mathbf{u}_h, \mathbf{v}^k_{h,p}) \\ - j_{\mathbf{u}_h}(\mathbf{u}_h, \mathbf{v}^k_{h,p}) + (\partial_t(\mathbf{u}-\mathbf{u}_h), \mathbf{v}^k_{h,p}).\end{aligned}$$

Proceeding term by term, in a fashion similar to the previous lemma with $\mathbf{v}^k_{h,p}$ instead of $\Pi_h^k \mathbf{v}_p$ and using an inverse inequality, we obtain

$$\begin{aligned}\|\Pi_h^k(\nabla p) - \Pi_h^k(\nabla p_h)\|^2_{0,\Omega} \leq Ch^{-2}\Big\{ h^{2r_p}\|p\|^2_{r_p,\Omega} \\ + h^{2r_\mathbf{u}-1}\|\mathbf{u}\|^2_{r_\mathbf{u},\Omega} + (1+h\|\mathbf{u}_h\|^2_{0,\infty,\Omega}) \|\!|\!|(\mathbf{u}-\mathbf{u}_h, p-p_h)|\!|\!|^2_{\mathbf{u}_h} \\ + \left(\|\mathbf{u}\|^2_{0,\infty,\Omega} + \|\mathbf{u}_h\|^2_{0,\infty,\Omega}\right)\|\mathbf{u}-\mathbf{u}_h\|^2_{0,\Omega}\Big\} \\ + C\left(h^{-1}\|\mathbf{u}_h\|^2_{0,\infty,\Omega}\|\!|\!|(\mathbf{u}-\mathbf{u}_h, p-p_h)|\!|\!|^2_{\mathbf{u}_h} + \|\partial_t(\mathbf{u}-\mathbf{u}_h)\|^2_{0,\Omega}\right).\end{aligned}$$

Finally, we conclude the proof after integration over $(0, T)$, application of the results of Corollaries 6.1 and 6.2 and (56). □

To close the problem of convergence of the pressure approximations we need an estimate of the error in the time derivative of the error. This is the subject of the next paragraph.

6.4 An estimate for $\partial_t(\mathbf{u} - \mathbf{u}_h)$

The following theorem states the main result of this paragraph.

Theorem 6.2 *Let $(\mathbf{u}, p)$ the solution of (1), $(\mathbf{u}_h, p_h) \in W_h^k$ the solution of (9), with $\mathbf{u}_{0,h} = P_h^k\mathbf{u}_0$, assume that $(\mathbf{u}, p)$ has the regularity (5)-(6) and that $\nu < h$. Then, the following estimate holds*

$$\int_0^T \|\partial_t(\mathbf{u}-\mathbf{u}_h)\|^2\,\mathrm{d}t \leq C\left(h^{2\alpha-3} + h^{2\alpha-1}\right), \tag{57}$$

with $\alpha \overset{\text{def}}{=} \min\{r_\mathbf{u}, r_p\}$ and $C > 0$ with no explicit dependence of ν and h.

Proof. We first decompose the error $(\mathbf{u}-\mathbf{u}_h, p-p_h)$ in two parts, using the projection operator $S_h \overset{\text{def}}{=} (P_h^k, R_h^k)$ defined by (34)

$$\begin{aligned}\mathbf{u}-\mathbf{u}_h &= \underbrace{\mathbf{u}-P_h^k\mathbf{u}}_{\boldsymbol{\theta}^\pi} + \underbrace{P_h^k\mathbf{u}-\mathbf{u}_h}_{\boldsymbol{\theta}_h} = \boldsymbol{\theta}^\pi + \boldsymbol{\theta}_h, \\ p-p_h &= \underbrace{p-R_h^k\mathbf{u}}_{y^\pi} + \underbrace{R_h^k\mathbf{u}-p_h}_{y_h} = y^\pi + y_h.\end{aligned} \tag{58}$$

Thus, using triangle inequality and from Corollary 4.2, we only need to estimate

$$\int_0^T \|\partial_t \boldsymbol{\theta}_h\|_{0,\Omega}^2 \, \mathrm{d}t.$$

To this aim, we first test the modified Galerkin orthogonality (Lemma 3.1) with $(\mathbf{v}_h, q_h) = (\partial_t \boldsymbol{\theta}_h, 0)$, to obtain

$$\begin{aligned}(\partial_t(\mathbf{u}-\mathbf{u}_h), \partial_t \boldsymbol{\theta}_h) + a_h(\mathbf{u}-\mathbf{u}_h, \partial_t \boldsymbol{\theta}_h) + b_h(p-p_h, \partial_t \boldsymbol{\theta}_h) \\ + c_h(\mathbf{u};\mathbf{u}, \partial_t \boldsymbol{\theta}_h) - c_h(\mathbf{u}_h;\mathbf{u}_h, \partial_t \boldsymbol{\theta}_h) \\ + \gamma j(\mathbf{u}-\mathbf{u}_h, \partial_t \boldsymbol{\theta}_h) + j_{\mathbf{u}_h}(\mathbf{u}-\mathbf{u}_h, \partial_t \boldsymbol{\theta}_h) = 0.\end{aligned}$$

Thus, using (58), we get

$$\begin{aligned}\|\partial_t \boldsymbol{\theta}_h\|_{0,\Omega}^2 + a_h(\boldsymbol{\theta}_h, \partial_t \boldsymbol{\theta}_h) + b_h(y_h, \partial_t \boldsymbol{\theta}_h) + \gamma j(\boldsymbol{\theta}_h, \partial_t \boldsymbol{\theta}_h) \\ = -(\partial_t \boldsymbol{\theta}^\pi, \partial_t \boldsymbol{\theta}_h) - a_h(\boldsymbol{\theta}^\pi, \partial_t \boldsymbol{\theta}_h) - b_h(y^\pi, \partial_t \boldsymbol{\theta}_h) + \gamma j(P_h^k \mathbf{u}, \partial_t \boldsymbol{\theta}_h) \\ + c_h(\mathbf{u}_h;\mathbf{u}_h, \partial_t \boldsymbol{\theta}_h) - c_h(\mathbf{u};\mathbf{u}, \partial_t \boldsymbol{\theta}_h) + j_{\mathbf{u}_h}(\mathbf{u}_h, \partial_t \boldsymbol{\theta}_h).\end{aligned} \tag{59}$$

Using the definition of a_h, from (12), one readily obtains that

$$a_h(\boldsymbol{\theta}_h, \partial_t \boldsymbol{\theta}_h) + \gamma j(\boldsymbol{\theta}_h, \partial_t \boldsymbol{\theta}_h) = \frac{1}{2}\partial_t \big[a_h(\boldsymbol{\theta}_h, \boldsymbol{\theta}_h) + \gamma j(\boldsymbol{\theta}_h, \boldsymbol{\theta}_h)\big]. \tag{60}$$

We now test (9) with $\mathbf{v}_h = \mathbf{0}$, and derive the remaining equation with respect to t, which yields

$$\begin{aligned}0 &= \partial_t(b_h(q_h, \mathbf{u}_h) - j(p_h, q_h)) \\ &= b_h(q_h, \partial_t \mathbf{u}_h) - j(\partial_t p_h, q_h) + b_h(\partial_t q_h, \mathbf{u}_h) - j(p_h, \partial_t q_h) \\ &= b_h(q_h, \partial_t \mathbf{u}_h) - j(\partial_t p_h, q_h).\end{aligned}$$

The last inequality is obtained by noticing that $\partial_t q_h \in V_h^k$ and therefore

$$b_h(\partial_t q_h, \mathbf{u}_h) - j(p_h, \partial_t q_h) = 0.$$

We then have

$$b_h(q_h, \partial_t \mathbf{u}_h) - j(\partial_t p_h, q_h) = 0, \tag{61}$$

for all $q_h \in V_h^k$. On the other hand, using the same argument, from (34) we obtain

$$b_h(q_h, \partial_t P_h^k \mathbf{u}) = j(\partial_t R_h^k \mathbf{u}, q_h), \tag{62}$$

for all $q_h \in V_h^k$. Thus, by taking $q_h = y_h$ in (61) and (62), we get

$$\begin{aligned}b_h(y_h, \partial_t \boldsymbol{\theta}_h) &= j(\partial_t y_h, y_h) \\ &= \frac{1}{2}\partial_t j(y_h, y_h).\end{aligned} \tag{63}$$

Therefore, by plugging (60) and (63) into (59), we have

$$
\begin{aligned}
&\|\partial_t \boldsymbol{\theta}_h\|_{0,\Omega}^2 + \frac{1}{2}\partial_t \Big[a_h(\boldsymbol{\theta}_h, \boldsymbol{\theta}_h) + j(y_h, y_h) + \gamma j(\boldsymbol{\theta}_h, \boldsymbol{\theta}_h) \Big] \\
&= \underbrace{-(\partial_t \boldsymbol{\theta}^\pi, \partial_t \boldsymbol{\theta}_h) - a_h(\boldsymbol{\theta}^\pi, \partial_t \boldsymbol{\theta}_h) - b_h(y^\pi, \partial_t \boldsymbol{\theta}_h) + \gamma j(P_h^k \mathbf{u}, \partial_t \boldsymbol{\theta}_h)}_{T_1} \\
&\quad + \underbrace{c_h(\mathbf{u}_h; \mathbf{u}_h, \partial_t \boldsymbol{\theta}_h) - c_h(\mathbf{u}; \mathbf{u}, \partial_t \boldsymbol{\theta}_h)}_{T_2} + \underbrace{j_{\mathbf{u}_h}(\mathbf{u}_h, \partial_t \boldsymbol{\theta}_h)}_{T_3}.
\end{aligned}
\tag{64}
$$

Now we estimate the terms T_i for $i = 1, \ldots, 3$. In the following, $\epsilon > 0$ stands for a constant to be fixed later on. For the first term, we use (34) to obtain,

$$
\begin{aligned}
T_1 &= -(\partial_t \boldsymbol{\theta}^\pi, \partial_t \boldsymbol{\theta}_h) + (\boldsymbol{\theta}^\pi, \partial_t \boldsymbol{\theta}_h) - b_h(p, \partial_t \boldsymbol{\theta}_h) \\
&\leq \frac{1}{2\epsilon}(\|\partial_t \boldsymbol{\theta}^\pi\|_{0,\Omega}^2 + \|\boldsymbol{\theta}^\pi\|_{0,\Omega}^2) + \frac{\epsilon}{2}\|\partial_t \boldsymbol{\theta}_h\|^2 - b_h(p, \partial_t \boldsymbol{\theta}_h).
\end{aligned}
\tag{65}
$$

The second term is treated as in (54), with $\partial_t \boldsymbol{\theta}_h$ in the place of $\Pi_h^k \mathbf{v}_p$. Therefore, after using an inverse inequality, we get

$$
\begin{aligned}
T_2 &\leq C h^{-1} \|\mathbf{u} - \mathbf{u}_h\|_{0,\Omega} \left(\|\mathbf{u}\|_{0,\infty,\Omega} + \|\mathbf{u}_h\|_{0,\infty,\Omega} \right) \|\nabla \partial_t \boldsymbol{\theta}_h\|_{0,\Omega} \\
&\quad + C h^{-\frac{1}{2}} \|\mathbf{u}_h\|_{0,\infty,\Omega} |||(\mathbf{u} - \mathbf{u}_h, p - p_h)|||_{\mathbf{u}_h} \|\partial_t \boldsymbol{\theta}_h\|_{0,\Omega}.
\end{aligned}
$$

Finally, using Cauchy-Schwarz and an inverse inequality

$$
\begin{aligned}
T_3 &\leq j_{\mathbf{u}_h}(\mathbf{u}_h, \mathbf{u}_h)^{\frac{1}{2}} j_{\mathbf{u}_h}(\partial_t \boldsymbol{\theta}_h, \partial_t \boldsymbol{\theta}_h)^{\frac{1}{2}} \\
&\leq C |||(\mathbf{u} - \mathbf{u}_h, p - p_h)|||_{\mathbf{u}_h} \|\mathbf{u}_h\|_{0,\infty,\Omega} h^{-\frac{1}{2}} \|\partial_t \boldsymbol{\theta}_h\|_{0,\Omega} \\
&\leq C \|\mathbf{u}_h\|_{0,\infty,\Omega}^2 \frac{h^{-1}}{2\epsilon} |||(\mathbf{u} - \mathbf{u}_h, p - p_h)|||_{\mathbf{u}_h}^2 + C \frac{\epsilon}{2} \|\partial_t \boldsymbol{\theta}_h\|_{0,\Omega}^2.
\end{aligned}
\tag{66}
$$

Integrating over $(0, T)$, using coercivity (Lemma 5.3) and by combining estimates (65)-(66) with Theorem 6.1, we obtain (for $\epsilon > 0$ sufficiently small)

$$
\begin{aligned}
\int_0^T \|\partial_t \boldsymbol{\theta}_h\|_{0,\Omega}^2 \,\mathrm{d}t &\leq C |||(\boldsymbol{\theta}_h(0), y_h(0))|||_{\mathbf{0}}^2 + C(\mathbf{u}, p, T) h^{2\min(r_{\mathbf{u}}, r_p) - 3} \\
&\quad + \int_0^T b_h(p, \partial_t \boldsymbol{\theta}_h) \,\mathrm{d}t.
\end{aligned}
\tag{67}
$$

First of all, we note that after partial integration first in space and then in time, we may write (for the last term on the right hand side),

$$\begin{aligned}\int_0^T b_h(p, \partial_t \boldsymbol{\theta}_h)\,\mathrm{d}t &= \int_0^T (\nabla p, \partial_t \boldsymbol{\theta}_h)\,\mathrm{d}t \\ &= -\int_0^T (\partial_t \nabla p, \boldsymbol{\theta}_h)\,\mathrm{d}t + (\nabla p(T), \boldsymbol{\theta}_h(T)) \\ &\quad - (\nabla p(0), \boldsymbol{\theta}_h(0)).\end{aligned}$$

By applying a Cauchy-Schwarz inequality and the Sobolev embedding

$$\|\nabla p\|_{L^\infty(0,T;L^2(\Omega))} \leq C\|\nabla p\|_{H^1(0,T;L^2(\Omega))},$$

in combination with Theorem 6.1 we have

$$\int_0^T b_h(p, \partial_t \boldsymbol{\theta}_h)\,\mathrm{d}t \leq C\|p\|_{H^1(0,T;H^1(\Omega))} h^{\min(r_{\mathbf{u}}, r_p)-\frac{1}{2}}.$$

Clearly, we have a triple norm contribution from the unknown initial discrete error in the pressure $y_h(0)$ in the right hand side of (67). Indeed, the term we need to control is

$$\begin{aligned}j(y_h(0), y_h(0)) &= j(R_h^k \mathbf{u}_0, R_h^k \mathbf{u}_0 - p_h(0)) \\ &\quad - j(p_h(0), R_h^k \mathbf{u}_0 - p_h(0)).\end{aligned} \tag{68}$$

Using the discrete incompressibility equations for $\mathbf{u}_h(0)$ and $P_h^k \mathbf{u}_0$ (and since $\nabla \cdot \mathbf{u}_0 = 0$) we have

$$\begin{aligned}b_h(q_h, \mathbf{u}_h(0)) &= j(p_h(0), q_h), \\ b_h(q_h, P_h^k \mathbf{u}_0) &= j(R_h^k \mathbf{u}_0, q_h).\end{aligned}$$

Thus, taking $q_h = R_h^k \mathbf{u}_0 - p_h(0)$ and since $\mathbf{u}_h(0) = P_h^k \mathbf{u}_0$, from (68) we have

$$\begin{aligned}0 &= b_h(R_h^k p(0) - p_h(0), P_h^k \mathbf{u}_0 - \mathbf{u}_h(0)) \\ &= j(R_h^k \mathbf{u}(0), R_h^k \mathbf{u}_0 - p_h(0)) - j(p_h(0), R_h^k \mathbf{u}_0 - p_h(0)) \\ &= j(y_h(0), y_h(0)).\end{aligned}$$

Hence we conclude that $j(y_h(0), y_h(0)) = 0$ and the theorem follows. $\square$

7 Conclusion

We have derived a priori error estimates for finite element approximations of the incompressible Navier-Stokes equations that are independent of the local Reynolds number and hence valid also for the incompressible Euler equations. The estimates are similar to those obtained in [23] in the case of piecewise linear elements and quasi optimal for the velocities, with the loss of $h^{\frac{1}{2}}$ with respect to approximation typical for stabilized methods.

For polynomial orders $k \geq 2$ the estimates for time derivative of the velocity of Theorem 6.2 is suboptimal in case the pressure is very regular due to the nonconsistency of the projection (34). The estimate can be improved if the analysis is performed in a time weighted norm and assuming "sufficient" regularity of the pressure. It is questionable if these stronger hypothesis can be justified (see the discussion in [25]). Convergence may also be proven assuming less regularity on the pressure, however if dependence on the viscosity is to be avoided it seems difficult to get away with less than $p \in H^1(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega))$. It is our hope that the present analysis sheds some light on the question of how to construct reliable numerical methods for large eddy simulation, where the local Reynolds number must always be assumed to be high. The fully discretized case and numerical examples will be addressed in a forthcoming work.

Acknowledgements

The first author acknowledges support by INRIA during his stay at Rocquencourt, in January 2005, as a visiting professor. The second author was partially supported by the Research Training Network "Mathematical Modelling of the Cardiovascular System (HaeMOdel)", contract HPRN-CT-2002-00270 of the European Community.

References

[1] R. Becker and M. Braack. A finite element pressure gradient stabilization for the Stokes equations based on local projections. *Calcolo*, 38(4):173–199, 2001.

[2] Roland Becker and Malte Braack. A two-level stabilization scheme for the Navier-Stokes equations. In *Numerical mathematics and advanced applications*, pages 123–130. Springer, Berlin, 2004.

[3] S. Bertoluzza. The discrete commutator property of approximation spaces. *C. R. Acad. Sci. Paris Sér. I Math.*, 329(12):1097–1102, 1999.

[4] Mats Boman. A posteriori error analysis in the maximum norm for a penalty finite element method for the time-dependent obstacle problem. Technical Report 2000-12, Chalmers Finite Element Center, 2000.

[5] M. Braack and E. Burman. Local projection stabilization for the oseen problem and its interpretation as a variational multiscale method. *SIAM J. Numer. Anal.*, 2005.

[6] J.H. Bramble, J.E. Pasciak, and O. Steinbach. On the stability of the L^2 projection in $H^1(\Omega)$. *Math. Comp.*, 71(237):147–156 (electronic), 2002.

[7] Susanne C. Brenner. Korn's inequalities for piecewise H^1 vector fields. *Math. Comp.*, 73(247):1067–1087 (electronic), 2004.

[8] F. Brezzi and M. Fortin. A minimal stabilisation procedure for mixed finite element methods. *Numer. Math.*, 89(3):457–491, 2001.

[9] E. Burman. A unified analysis for conforming and nonconforming stabilized finite element methods using interior penalty. *SIAM J. Numer. Anal.*, 2004. in press.

[10] E. Burman, M.A. Fernández, and P. Hansbo. Edge stabilization for the incompressible Navier-Stokes equations: a continuous interior penalty finite element method. Technical Report RR-5349, INRIA, 2004. Submitted.

[11] Erik Burman and Peter Hansbo. Edge stabilization for Galerkin approximations of convection-diffusion-reaction problems. *Comput. Methods Appl. Mech. Engrg.*, 193(15-16):1437–1453, 2004.

[12] Erik Burman and Peter Hansbo. Edge stabilization for the generalized Stokes problem: a continuous interior penalty method. *Comput. Methods Appl. Mech. Engrg.*, 2005. in press.

[13] Ph. Clément. Approximation by finite element functions using local regularization. *Rev. Française Automat. Informat. Recherche Opérationnelle Sér. Rouge Anal. Numér.*, 9(R-2):77–84, 1975.

[14] Ramon Codina. Stabilized finite element approximation of transient incompressible flows using orthogonal subscales. *Comput. Methods Appl. Mech. Engrg.*, 191(39-40):4295–4321, 2002.

[15] Ramon Codina and Jordi Blasco. Analysis of a pressure-stabilized finite element approximation of the stationary Navier-Stokes equations. *Numer. Math.*, 87(1):59–81, 2000.

[16] M. Crouzeix and V. Thomée. The stability in L_p and W_p^1 of the L_2-projection onto finite element function spaces. *Math. Comp.*, 48(178):521–532, 1987.

[17] J. Douglas Jr. and T. Dupont. *Interior Penalty Procedures for Elliptic and Parabolic Galerkin Methods*, volume 58 of *Lecture Notes in Physics*. Springer-Verlag, Berlin, 1976.

[18] A. Ern and J.-L. Guermond. *Theory and practice of finite elements*, volume 159 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 2004.

[19] J. Freund and R. Stenberg. On weakly imposed boundary conditions for second order problems. In M. Morandi Cecchi et al., editor, *Proceedings of the Ninth Int. Conf. Finite Elements in Fluids*, pages 327–336, Venice, 1995.

[20] V. Girault and P.-A. Raviart. *Finite element methods for Navier-Stokes equations*, volume 5 of *Springer Series in Computational Mathematics*. Springer-Verlag, Berlin, 1986.

[21] Jean-Luc Guermond. Stabilization of Galerkin approximations of transport equations by subgrid modeling. *M2AN Math. Model. Numer. Anal.*, 33(6):1293–1316, 1999.

[22] Jean-Luc Guermond and Serge Prudhomme. Mathematical analysis of a spectral hyperviscosity LES model for the simulation of turbulent flows. *M2AN Math. Model. Numer. Anal.*, 37(6):893–908, 2003.

[23] Peter Hansbo and Anders Szepessy. A velocity-pressure streamline diffusion finite element method for the incompressible Navier-Stokes equations. *Comput. Methods Appl. Mech. Engrg.*, 84(2):175–192, 1990.

[24] Y. He, Y. Lin, and W. Sun. Stabilized finite element method for the non-stationnary navier-stokes problem. *Preprint*, 2004.

[25] John G. Heywood and Rolf Rannacher. Finite element approximation of the nonstationary Navier-Stokes problem. I. Regularity of solutions and second-order error estimates for spatial discretization. *SIAM J. Numer. Anal.*, 19(2):275–311, 1982.

[26] John G. Heywood and Rolf Rannacher. Finite element approximation of the nonstationary Navier-Stokes problem. II. Stability of solutions and error estimates uniform in time. *SIAM J. Numer. Anal.*, 23(4):750–777, 1986.

[27] John G. Heywood and Rolf Rannacher. Finite element approximation of the nonstationary Navier-Stokes problem. III. Smoothing property and higher order error estimates for spatial discretization. *SIAM J. Numer. Anal.*, 25(3):489–512, 1988.

[28] R.H.W. Hoppe and B. Wohlmuth. Element-oriented and edge-oriented local error estimators for nonconforming finite element methods. *RAIRO Modél. Math. Anal. Numér.*, 30(2):237–263, 1996.

[29] Claes Johnson and Jukka Saranen. Streamline diffusion methods for the incompressible Euler and Navier-Stokes equations. *Math. Comp.*, 47(175):1–18, 1986.

[30] P. Oswald. On a BPX-preconditioner for P1 elements. *Computing*, 51(2):125–133, 1993.

[31] L. Ridgway Scott and Shangyou Zhang. Finite element interpolation of nonsmooth functions satisfying boundary conditions. *Math. Comp.*, 54(190):483–493, 1990.

[32] Lutz Tobiska and Rüdiger Verfürth. Analysis of a streamline diffusion finite element method for the Stokes and Navier-Stokes equations. *SIAM J. Numer. Anal.*, 33(1):107–127, 1996.

Unité de recherche INRIA Rocquencourt
Domaine de Voluceau - Rocquencourt - BP 105 - 78153 Le Chesnay Cedex (France)

Unité de recherche INRIA Futurs : Parc Club Orsay Université - ZAC des Vignes
4, rue Jacques Monod - 91893 ORSAY Cedex (France)
Unité de recherche INRIA Lorraine : LORIA, Technopôle de Nancy-Brabois - Campus scientifique
615, rue du Jardin Botanique - BP 101 - 54602 Villers-lès-Nancy Cedex (France)
Unité de recherche INRIA Rennes : IRISA, Campus universitaire de Beaulieu - 35042 Rennes Cedex (France)
Unité de recherche INRIA Rhône-Alpes : 655, avenue de l'Europe - 38334 Montbonnot Saint-Ismier (France)
Unité de recherche INRIA Sophia Antipolis : 2004, route des Lucioles - BP 93 - 06902 Sophia Antipolis Cedex (France)

Éditeur
INRIA - Domaine de Voluceau - Rocquencourt, BP 105 - 78153 Le Chesnay Cedex (France)
http://www.inria.fr
ISSN 0249-6399